TAMELY RAMIFIED SUBFIELDS OF DIVISION ALGEBRAS

DANNY NEFTIN

ABSTRACT. For any number field K, it is unknown which finite groups appear as Galois groups of extensions L/K such that L is a maximal subfield of a division algebra with center K (a K-division algebra). For $K = \mathbb{Q}$, the answer is described by the long standing $\mathbb{Q}$-admissibility conjecture.

We extend a theorem of Neukirch on embedding problems with local constraints in order to determine for every number field K, what finite solvable groups G appear as Galois groups of tame maximal subfields of K-division algebras, generalizing Liedahl's theorem for metacyclic G and Sonn's solution of the $\mathbb{Q}$-admissibility conjecture for solvable groups.

1. INTRODUCTION

A division algebra D which is finite dimensional over its center K (*a K-division algebra*), is called a *G-crossed product* if there exists a Galois extension L/K with Galois group G (a *G-extension*) such that L is a maximal subfield of D. Crossed products are fundamental in the study of division algebras and are accompanied by a structure which explicitly describes them (see [20, Chp. 14-19]). A group G is called *K-admissible* if there exists a G-crossed product K-division algebra; a field extension L/K is called *adequate* if L is a maximal subfield of a K-division algebra [1].

It is known by the Brauer-Hasse-Noether theorem that over a number field K, all K-division algebras are crossed products with respect to a cyclic group. However, it is unknown for which groups G there exists a G-crossed product K-division algebra, i.e. what groups are K-admissible?

Over $\mathbb{Q}$, Schacher observed ([21]) that the Sylow subgroups P of a $\mathbb{Q}$-admissible group are *metacyclic*, that is P has a cyclic normal subgroup $C \lhd P$ such that P/C is also cyclic. The converse of this observation is known as the $\mathbb{Q}$-admissibility conjecture:

Conjecture 1.1. *Every group with metacyclic Sylow subgroups is $\mathbb{Q}$-admissible.*

[1] In fact by [21], L/K is adequate if and only if L is a subfield of a K-division algebra. Thus, the maximality requirement can be omitted.

This conjecture was studied extensively (e.g. [4],[5],[6],[10],[11],[21]) and proven by Sonn for solvable groups in a series of papers ([3], [23] and [24]).

Recently, analogs of this conjecture were proved by Harbater, Hartmann and Krashen over function fields of curves over complete discretely valued fields with algebraically closed residue fields ([13], cf. [12]), by Paran and the author over two dimensional complete local domains with algebraically closed residue fields ([16]), and by Surendranath and Suresh over function fields of curves over complete discretely valued fields which contain enough roots of unity ([25]). However, the situation over number fields is far from being understood.

Schacher's observation extends to number fields under an additional assumption of tameness as follows. Let μ_n denote the set of n-th roots of unity and $\sigma_{t,n}$ be the automorphism of $\mathbb{Q}(\mu_n)$ for which $\sigma_{t,n}(\zeta) = \zeta^t$ for all $\zeta \in \mu_n$. Using a similar argument to Liedahl's [14, Theorem 28], we observe that if G appears as a Galois group of a tamely ramified adequate extension of a number field K then its Sylow subgroups are metacyclic, and furthermore for every $l \,|\, |G|$, the l-Sylow subgroups $G(l)$ of G admit a presentation:

$$G(l) \cong \mathcal{M}(m,n,i,t) := \langle x, y | x^m = y^i, y^n = 1, x^{-1}yx = y^t\rangle \tag{1.1}$$

such that $\sigma_{t,n}$ fixes $K \cap \mathbb{Q}(\mu_n)$ (see "only if part" of Theorem 1.3).

This observation suggests the following natural generalization of Conjecture 1.1:

Question 1.2. Let K be a number field and G a group whose l-Sylow subgroups admit a presentation $\mathcal{M}(m,n,i,t)$ such that $\sigma_{t,n}$ fixes $K \cap \mathbb{Q}(\mu_n)$, for every $l \,|\, |G|$. Is G necessarily K-admissible? Furthermore, is there necessarily a tamely ramified adequate G-extension of K?

The first part of this question is known to have an affirmative answer for metacyclic G ([14, Theorem 27]) and for some small order groups: A_5 ([11]), the central extension $\mathrm{SL}_2(5)$ of A_5 ([9]), A_6, A_7 ([22]), the double covers of A_6 and A_7 ([8]), $\mathrm{PSL}_2(7)$ ([1]) and $\mathrm{PSL}_2(11)$ ([7]).

In this paper we give a positive answer to Question 1.2 for solvable groups, generalizing Liedahl's [14, Theorem 27] and Sonn's [24, Theorem 1]:

Theorem 1.3. *Let K be a number field and G a solvable group. Then there exists a tamely ramified adequate G-extension L/K if and only if for every $l||G|$, the l-Sylow subgroups of G admit a presentation $\mathcal{M}(m,n,i,t)$ such that $\sigma_{t,n}$ fixes $K \cap \mathbb{Q}(\mu_n)$.*

We note that since the proof of Sonn's theorem ([24]) over $\mathbb{Q}$ is based on Neukirch's [17, Main Theorem] which makes an assumption on the absence of roots of unity in K, Sonn's proof does not apply over arbitrary number fields.

A key ingredient in our proof is an extension of [18, Korollar 6.4]. Neukirch's Korollar 6.4 is a highly useful tool that under the assumption of at least one of six conditions on a finite set S of primes of the base field, allows to change solutions of embedding problems to satisfy any prescribed local conditions at S (generalizing the Grunwald-Wang theorem). We extend Korollar 6.4 by showing that under the assumption of at least one of four of these six conditions on S, it is possible to change a solution to satisfy prescribed conditions at S leaving the solution unchanged at any given finite set of primes T.

We use this extension to strengthen Sonn's proof of [24, Theorem 1] in order to obtain tamely ramified adequate G-extensions of $\mathbb{Q}$ with prescribed local behavior at given finite sets of primes. This gives us a strong control over the ramification of G-crossed product $\mathbb{Q}$-division algebras, allowing us to lift these to division algebras over a given number field and by that prove Theorem 1.3.

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2. Embedding problems and local Galois groups

2.1. **Embedding problems.**

The theory of embedding problems is central in the study of the inverse Galois problem and is a key ingredient in our proof of Theorem 1.3. We shall describe a setup for these problems, recall Neukirch's [18, Korollar 6.4] and extend it.

2.1.1. *Setup.* Embedding problems are a strong generalization of the inverse Galois problem which ask whether a Galois extension can be embedded into a larger Galois extension with a given Galois group. The precise setup is as follows.

A *(finite) embedding problem* over a number field K consists of a finite Galois extension L/K and an epimorphism of finite groups $\pi : E \to G := \mathrm{Gal}(L/K)$. For our purposes it suffices to consider embedding problems with abelian kernel $A := \ker(\pi)$.

Let G_K denote the absolute Galois group of K. Two homomorphisms $\psi_1, \psi_2 : G_K \to E$ are called equivalent if there is an $a \in A$ such that $a^{-1}\psi_1(g)a = \psi_2(g)$ for all $g \in G_K$. A *solution* for π is an equivalence class of homomorphisms $\psi : G_K \to E$ (not necessarily surjective) for which $\pi \circ \psi$ is the restriction map $\mathrm{res}_L : G_K \to G$. For a surjective solution ψ, the fixed field $M = \overline{K}^{\ker(\psi)}$ contains L and has Galois group $\mathrm{Gal}(M/K) \cong E$.

The epimorphism π defines an action of G on A and hence induces a G_K-module structure on A via res_L. For every crossed homomorphism $\chi \in \mathrm{H}^1(G_K, A)$ and solution $\psi : G_K \to E$, the map $\psi' = \chi \cdot \psi$ given by $\psi'(\sigma) = \chi(\sigma)\psi(\sigma)$ for all $\sigma \in G_K$, is also a solution (see [19, Chp.

IX, §4]). In fact, for every two solutions ψ, ψ' of π, there is a unique $\chi \in \mathrm{H}^1(G_K, A)$ such that $\psi' = \chi \cdot \psi$. We think of χ as the element that "changes" ψ to ψ'.

2.1.2. *Embedding problems with prescribed local conditions.* By a prime $\mathfrak{p}$ of K we mean a finite or infinite prime. Fix an algebraic closure $\overline{K}$ of K, an algebraic closure $\overline{K}_\mathfrak{p}$ of the completion $K_\mathfrak{p}$, and an inclusion of $\overline{K}$ into $\overline{K}_\mathfrak{p}$ for every prime $\mathfrak{p}$ of K. In particular, the embedding problem π induces a local embedding problem $\pi_\mathfrak{p} : \pi^{-1}(G_\mathfrak{p}) \to G_\mathfrak{p}$ where $G_\mathfrak{p} = \mathrm{Gal}(L_\mathfrak{p}/K_\mathfrak{p}), L_\mathfrak{p} := LK_\mathfrak{p}$. Moreover, the restriction $\psi_\mathfrak{p}$ of a solution $\psi : G_K \to E$ to the subgroup $G_{K_\mathfrak{p}}$ is a solution for $\pi_\mathfrak{p}$.

Let S be a finite set of primes of K and for every $\mathfrak{p} \in S$ fix (prescribe) a solution $\psi^{(\mathfrak{p})}$ to $\pi_\mathfrak{p}$, assuming such (local) solutions exist. Similarly to the Grunwald-Wang theorem, one is interested in solutions ψ of π such that $\psi_\mathfrak{p} = \psi^{(\mathfrak{p})}$ for all $\mathfrak{p} \in S$.

Assume π has a solution ϕ. Then for every $\mathfrak{p} \in S$ there is $\chi^{(\mathfrak{p})} \in \mathrm{H}^1(G_{K_\mathfrak{p}}, A)$ such that $\psi^{(\mathfrak{p})} = \chi^{(\mathfrak{p})} \cdot \phi_\mathfrak{p}$. If the element $(\chi^{(\mathfrak{p})})_{\mathfrak{p}\in S}$ has a source χ under the restriction map:

$$\rho_S : \mathrm{H}^1(G_K, A) \to \prod_{\mathfrak{p}\in S} \mathrm{H}^1(G_{K_\mathfrak{p}}, A)$$

then $\psi := \chi \cdot \phi$ is a solution for π which restricts to $\psi^{(\mathfrak{p})} = \chi^{(\mathfrak{p})} \cdot \phi_\mathfrak{p}$ at all $\mathfrak{p} \in S$. Thus, if the map ρ_S is surjective, every solution for π can be "changed" to a solution with prescribed local conditions at S.

2.1.3. *Neukirch's Korollar.* [18, Korollar 6.4] is a highly useful criteria for the map ρ_S to be surjective. Let A be a G_K-module and $n = \exp(A)$. Let $A' = \mathrm{Hom}(A, \mu_n)$ be the dual G_K-module and $K(A')$ the fixed field of the centralizer of A' in G_K. Let $G' := \mathrm{Gal}(K(A')/K)$ and for a prime $\mathfrak{p}$ of K, let $G'_\mathfrak{p} := \mathrm{Gal}(K(A')_\mathfrak{p}/K_\mathfrak{p})$. Denote $\Gamma(G, A) := \ker\left(\mathrm{H}^1(G, A) \to \prod_{g\in G} \mathrm{H}^1(\langle g\rangle, A)\right)$.

Theorem 2.1. (Neukirch [18, Korollar 6.4]) *Let S be a finite set of primes of K. Then the map ρ_S is surjective in each of the following cases:*

(a) $\Gamma(G'_\mathfrak{p}, A') = 0$ *for all* $\mathfrak{p} \in S$,
(b) *for every $\mathfrak{p} \in S$, the group $G'_\mathfrak{p}$ is cyclic or a semidirect product of two cyclic groups of relatively prime orders,*
(c) $\mathrm{H}^1(G', A') = 0$,
(d) $|G'| = \mathrm{lcm}\{|G'_\mathfrak{p}| | \mathfrak{p} \notin S\}$,
(e) *A is cyclic of odd order,*
(f) *the action of G_K on A is trivial and $(K, \exp(A), S)$ does not fall into a special case.*

In (f), *when* $\exp(A) = 2^t m$, *m odd, one says that the triple* $(K, \exp(A), S)$ *falls into a special case if $K(\mu_{2^t})/K$ is noncyclic and S contains all primes $\mathfrak{p}$ for which $K_\mathfrak{p}(\mu_{2^t})/K_\mathfrak{p}$ is noncyclic.*

Thus, under each of these conditions one can change a solution to satisfy arbitrary prescribed local conditions at S. Furthermore, we show that under each of conditions (a), (b), (c) or (e) it is possible to change a solution to satisfy prescribed local conditions at S leaving the solution unchanged at a given finite set of primes T.

Proposition 2.2. *Let A be a finite G_K-module. Assume that conditions (a) or (b) hold for a finite set S. Then the subgroup*

$$\prod_{\mathfrak{p}\in S} \mathrm{H}^1(G_{K_\mathfrak{p}}, A) \times \prod_{\mathfrak{p}\in T}\{0\}$$

is in the image of $\rho_{S\cup T}$ for every finite set T disjoint from S.

Proof. Since by [18, Satz 6.2] condition (b) implies (a), it suffices to prove the assertion when (a) holds. Assume that $\Gamma(G'_\mathfrak{p}, A') = 0$ for all $\mathfrak{p} \in S$. Let P be the set of all primes of K and $\prod'_{\mathfrak{p}\in P} \mathrm{H}^1(G_{K_\mathfrak{p}}, A)$ the restricted product over the subgroup $\prod_{\mathfrak{p}\in P} \mathrm{H}^1_{un}(G_{K_\mathfrak{p}}, A)$. Recall that the Poitou-Tate theorem gives a non-degenerate bilinear map

$$\beta : \prod_{\mathfrak{p}\in P}{}' \mathrm{H}^1(G_{K_\mathfrak{p}}, A) \times \prod_{\mathfrak{p}\in P}{}' \mathrm{H}^1(G_{K_\mathfrak{p}}, A') \to \mathbb{Q}/\mathbb{Z}$$

which is defined as the product of local bilinear maps

$$\beta_\mathfrak{p} : \mathrm{H}^1(G_{K_\mathfrak{p}}, A) \times \mathrm{H}^1(G_{K_\mathfrak{p}}, A') \to \mathbb{Q}/\mathbb{Z}$$

for every $\mathfrak{p} \in P$.

Following [18], for a finite set U of primes of K we let:

$$\rho'_U : \mathrm{H}^1(G_K, A') \to \prod_{\mathfrak{p}\notin U}{}' \mathrm{H}^1(G_{K_\mathfrak{p}}, A')$$

be the restriction map, $\Delta = \mathrm{coker}(\rho_{S\cup T})$ and $\nabla = \ker(\rho'_{S\cup T})/\ker(\rho'_\emptyset)$. By [18, Satz 4.4], β induces a non-degenerate bilinear form $\beta_0 : \Delta \times \nabla \to \mathbb{Q}/\mathbb{Z}$, which is given on $\chi := (\chi_\mathfrak{p})_{\mathfrak{p}\in S\cup T} \in \Delta$ and $\lambda \in \nabla$ by $\beta_0(\chi, \lambda) := \beta(\tilde{\chi}, \rho'_\emptyset(\lambda))$ where $\tilde{\chi} \in \prod'_{\mathfrak{p}\in P} \mathrm{H}^1(G_{K_\mathfrak{p}}, A)$ is any element whose $\mathfrak{p}$-th component is $\chi_\mathfrak{p}$ at all $\mathfrak{p} \in S \cup T$.

Let $\chi = \prod_{\mathfrak{p}\in S\cup T} \chi_\mathfrak{p}$ be an element of Δ such that $\chi_\mathfrak{p} = 0$ for all $\mathfrak{p} \in T$. We claim that χ is orthogonal to ∇ and therefore it is the zero element in Δ, proving the proposition.

Letting $\tilde{\chi} = (\tilde{\chi}_\mathfrak{p})_{\mathfrak{p}\in P} \in \prod'_{\mathfrak{p}\in P} \mathrm{H}^1(G_{K_\mathfrak{p}}, A)$ where $\tilde{\chi}_\mathfrak{p} = \chi_\mathfrak{p}$ for $\mathfrak{p} \in S$ and $\tilde{\chi}_\mathfrak{p} = 0$ for $\mathfrak{p} \notin S$, we have $\beta_0(\chi, \nabla) = \beta(\tilde{\chi}, \rho'_\emptyset(\nabla))$. Since $\tilde{\chi}_\mathfrak{p} = 0$ for $\mathfrak{p} \notin S$, it suffices to show that $\beta_\mathfrak{p}(\chi_\mathfrak{p}, \rho'_\emptyset(\nabla)_\mathfrak{p}) = 0$ for all $\mathfrak{p} \in S$, where $\rho'_\emptyset(\nabla)_\mathfrak{p}$ is the projection of $\rho'_\emptyset(\nabla)$ to the $\mathfrak{p}$-th factor. But [18, Satz 6.3] implies that the image of ∇ under the restriction map

$$\rho_{S\cup T, A'} : \mathrm{H}^1(G_K, A') \to \prod_{\mathfrak{p}\in S\cup T} \mathrm{H}^1(G_{K_\mathfrak{p}}, A')$$

lies in $\prod_{\mathfrak{p}\in S\cup T}\Gamma(G'_\mathfrak{p},A')$. Since by assumption $\Gamma(G'_\mathfrak{p},A')=0$ for $\mathfrak{p}\in S$, we get $\rho'_\emptyset(\nabla)_\mathfrak{p}=\rho_{S\cup T,A'}(\nabla)_\mathfrak{p}=0$ and hence $\beta_\mathfrak{p}(\chi_\mathfrak{p},\pi_\mathfrak{p}\rho'_\emptyset(\nabla))=0$ for all $\mathfrak{p}\in S$, proving the claim.

□

From Proposition 2.2 and the discussion above it we get:

Corollary 2.3. *Let $\pi : E\to \mathrm{Gal}(L/K)$ be an embedding problem with solution ϕ. Fix solutions $\psi^{(\mathfrak{p})}$ for $\pi_\mathfrak{p}$ at all primes $\mathfrak{p}$ in a finite set S and let T be a finite set of primes disjoint from S. Assume that at least one of conditions* (a),(b),(c) *or* (e) *hold for S.*

Then there exists a solution ψ such that $\psi_\mathfrak{p}=\psi^{(\mathfrak{p})}$ for all $\mathfrak{p}\in S$ and $\psi_\mathfrak{p}=\phi_\mathfrak{p}$ for all $\mathfrak{p}\in T$.

Proof. Since conditions (c) and (e) are independent of S, the image of $\rho_{S\cup T}$ contains $\prod_{\mathfrak{p}\in S}\mathrm{H}^1(G_{K_\mathfrak{p}},A)\times\prod_{\mathfrak{p}\in T}\{0\}$ under these conditions as well. For $\mathfrak{p}\in S$, let $\chi^{(\mathfrak{p})}\in\mathrm{H}^1(G_{K_\mathfrak{p}},A)$ be the element for which $\psi^{(\mathfrak{p})}=\chi^{(\mathfrak{p})}\cdot\psi_\mathfrak{p}$. By Proposition 2.2, the element $(\chi^{(\mathfrak{p})})_{\mathfrak{p}\in S}\times(0)_{\mathfrak{p}\in T}$ has a source $\chi\in\mathrm{H}^1(G_K,A)$ under the map $\rho_{S\cup T}$. Then the solution $\psi:=\chi\cdot\phi$ restricts to $\chi^{(\mathfrak{p})}\cdot\phi_\mathfrak{p}=\psi^{(\mathfrak{p})}$ at all $\mathfrak{p}\in S$ and to $0\cdot\phi_\mathfrak{p}=\phi_\mathfrak{p}$ at all $\mathfrak{p}\in T$. □

Remark 2.4.
1. Proposition 2.2 need not hold under conditions (d) or (f). For example, let K be a quadratic extension of $\mathbb{Q}$ in which 2 splits and let $\mathfrak{p}_1,\mathfrak{p}_2$ be the primes above it. Let $S=\{\mathfrak{p}_1\}$, $T=\{\mathfrak{p}_2\}$ and let $A=\mathbb{Z}/8$ be the trivial G_K-module. Then $A'\cong\mu_8$ as G_K-modules and $K(A')=K(\mu_8)$. Both conditions (d) and (f) hold for S and hence ρ_S is surjective.

   However, since $K(A')_\mathfrak{p}/K_\mathfrak{p}$ is cyclic for all $\mathfrak{p}\neq\mathfrak{p}_1,\mathfrak{p}_2$, conditions (d) and (f) fail for $S\cup T$. Furthermore, the Grunwald-Wang theorem shows that
   $$\prod_{\mathfrak{p}\in S}\mathrm{H}^1(G_{K_\mathfrak{p}},A)\times\prod_{\mathfrak{p}\in T}\{0\}\not\subseteq\mathrm{Im}\,\rho_{S\cup T}.$$
   Indeed, letting $\psi^{(\mathfrak{p}_2)}=0$ and $\psi^{(\mathfrak{p}_1)}\in\mathrm{Hom}(G_{K_{\mathfrak{p}_1}},A)$ be such that the fixed field of $\ker(\psi^{(\mathfrak{p}_1)})$ is the unramified $\mathbb{Z}/8$-extension of K, [2, Chp. X, Theorem 5] shows that $(\psi^{(\mathfrak{p}_1)},\psi^{(\mathfrak{p}_2)})\notin\mathrm{Im}(\rho_{S\cup T})$.
2. Let A be a trivial G_K-module of exponent $2^tm'$ where m' is odd. If $K(\mu_{2^t})/K$ is cyclic then condition (f) holds for all finite sets S.

2.2. **Tame Galois groups of local fields.** We shall make use of a few well known facts about Galois groups of tame local extensions, all of which can be found in [26] and [19, Chp. VII, §5]. Let L/K a tamely ramified G-extension of p-adic fields, I its inertia group, $n:=|I|$, and q the cardinality of the residue field of K.

The subfield L^I contains μ_n and L/L^I is a (cyclic) Kummer extension. The Galois group of L^I/K is generated by the Frobenius automorphism σ_L which acts on μ_n by raising each element to the power q. In particular the restriction of σ_L to $\mathbb{Q}(\mu_n)$ is $\sigma_{q,n}$ and fixes $\mathbb{Q}(\mu_n) \cap K$. The action of σ on I via conjugation in G is equivalent to its action on μ_n. Thus,

$$G = \mathcal{M}(m, n, i, t) = \langle x, y | x^m = y^i, y^n = 1, x^{-1}yx = y^t \rangle,$$

where $t \equiv q \pmod n$, $I = \langle y \rangle$ and $x = \sigma \pmod I$. In particular, one has the following observation which is the basis to [14, Theorem 28]:

Lemma 2.5. *Let p, l be two distinct rational primes and K a p-adic field. Then every group G that appears as a Galois group over K has a metacyclic l-Sylow subgroup $\mathcal{M}(m, n, i, t)$ such that $\sigma_{t,n}$ fixes $K \cap \mathbb{Q}(\mu_n)$.*

Proof. Let L/K be a G-extension, M the fixed subfield of an l-Sylow subgroup H of G and t the cardinality of the residue field of M. Then $H \cong \mathcal{M}(m, n, i, t)$ and $\sigma_{t,n}$ fixes $M \cap \mathbb{Q}(\mu_n)$. In particular $\sigma_{t,n}$ fixes $K \cap \mathbb{Q}(\mu_n)$. □

Consider the converse problem of realizing $\mathcal{M}(m, n, i, t)$ over K and assume $t \equiv q \pmod n$ so that $\mathcal{M}(m, n, i, t) = \mathcal{M}(m, n, i, q)$. The Galois group G_K^{tr} of the maximal tamely ramified extension of K is profinitely generated by two automorphisms σ and τ and one relation $\sigma^{-1}\tau\sigma = \tau^q$, where τ is of order prime to q and σ is the Frobenius automorphism. Letting M be the (unique) unramified degree m extension of K, σ restricts to the Frobenius automorphism of M/K. Thus, an embedding problem $\pi : \mathcal{M}(m, n, i, t) \to \mathrm{Gal}(M/K)$ with kernel $\langle y \rangle$ has a surjective solution whose corresponding field is a tamely ramified $\mathcal{M}(m, n, i, t)$-extension of K.

3. Galois groups of tamely ramified adequate extensions

3.1. **Proof of Theorem 1.3.** We consider a refined notion of adequacy. For a number field K and a finite set S of primes of K, we say that L/K is *S-adequate* if L is a maximal subfield of a K-division algebra that is unramified outside S. Let $D(L/K, \mathfrak{P})$ denote the decomposition group of a prime $\mathfrak{P}$ of L. The same proof as of Schacher's criterion ([21, Proposition 2.6]) gives the following criterion for S-adequacy:

Proposition 3.1. *Let L/K be a G-extension of number fields and S a finite set of primes of K. Then L/K is S-adequate if and only if for every rational prime $l \mid |G|$, there are two primes $\mathfrak{p}_1, \mathfrak{p}_2 \in S$ for which $D(L/K, \mathfrak{P}_i)$ contains an l-Sylow subgroup of G, where $\mathfrak{P}_i$ is a prime of L which divides $\mathfrak{p}_i$, $i = 1, 2$.*

Note that the condition of containing an l-Sylow subgroup of G is independent of the choice of prime $\mathfrak{P}_i$ dividing $\mathfrak{p}_i$.

A key ingredient in our proof of Theorem 1.3 is the following generalization of Sonn's theorem ([24, Theorem 1]) which asserts the existence of S-adequate G-extensions for prescribed sets S. Since $\mathcal{M}(m,n,i,t)$ is realizable over $\mathbb{Q}_p$ when $p \equiv t \pmod{n}$ (see Section 2.2), we consider the following sets S:

Definition 3.2. We call a set S of distinct odd rational primes $p_i^{(l)}, i = 1,2, l||G|$ which are prime to $|G|$, a *tame supporting set for G* if for every $l||G|$, the l-Sylow subgroups of G admit a presentation $\mathcal{M}(m,n,i,t)$ such that $p_1^{(l)}, p_2^{(l)} \equiv t \pmod{n}$.

Theorem 3.3. *Let G be a solvable group with metacyclic Sylow subgroups. Let S be a tame supporting set for G and T a finite set of rational primes which is disjoint from S. Then there exists an S-adequate G-extension $L/\mathbb{Q}$ in which the primes of T split completely.*

The proof of this theorem is based on Corollary 2.3 and on Sonn's proof of [24], and is given in Section 4. We now use this theorem to prove Theorem 1.3:

Proof of Theorem 1.3. "Only if part": Let l be a prime that divides $|G|$. By Proposition 3.1, there is a prime $\mathfrak{p}$ of K such that $D(L/K,\mathfrak{P}), \mathfrak{P} \mid \mathfrak{p}$, contains an l-Sylow subgroup P of G. If $\mathfrak{p} \mid l$, $\mathfrak{p}$ is unramified in L and hence P is cyclic. Otherwise, Lemma 2.5 implies that P has a presentation $\mathcal{M}(m,n,i,t)$ such that $\sigma_{t,n}$ fixes $K_{\mathfrak{p}} \cap \mathbb{Q}(\mu_n)$ and hence $K \cap \mathbb{Q}(\mu_n)$. In both cases, P has the required presentation.

"If part": Let $l \mid |G|$ be a rational prime and let $P^{(l)}$ be the set of primes of K that are unramified over $\mathbb{Q}$ with residue degree 1, and whose restriction p to $\mathbb{Q}$ satisfies $p \equiv t_l \pmod{n_l}$.

We first claim that $P^{(l)}$ is infinite. Let M be the $\mathbb{Q}$-normal closure of K. Since σ_{t_l,n_l} fixes $K \cap \mathbb{Q}(\mu_{n_l})$, σ_{t_l,n_l} extends to an automorphism of $K(\mu_{n_l})$ that fixes K and hence lifts to an automorphism $\tau_l \in \mathrm{Gal}(M(\mu_{n_l})/K)$. By Chebotarev's density theorem there are infinitely many primes $\mathfrak{P}$ of $M(\mu_{n_l})$ that are unramified over $\mathbb{Q}$, and whose Frobenius automorphism in $M(\mu_{n_l})/\mathbb{Q}$ is τ_l. In particular, the restriction p of such $\mathfrak{P}$ to $\mathbb{Q}$ has Frobenius σ_{t_l,n_l} in $\mathbb{Q}(\mu_{n_l})/\mathbb{Q}$, and hence $p \equiv t_l \pmod{n_l}$. Since τ_l fixes K, the restriction of each such $\mathfrak{P}$ to K has residue degree 1 over $\mathbb{Q}$ and hence is in $P^{(l)}$, proving the claim.

Since $P^{(l)}$ is infinite, we can choose two primes $\mathfrak{p}_1^{(l)}, \mathfrak{p}_2^{(l)} \in P^{(l)}$ such that the restrictions $p_i^{(l)}$ of $\mathfrak{p}_i^{(l)}$ to $\mathbb{Q}$, $i = 1,2, l \mid |G|$, are all distinct rational primes which are prime to $|G|$. Thus, the set $S := \{p_i^{(l)} | i = 1,2, l||G|\}$ is a tame supporting set for G and by Theorem 3.3 there exists an S-adequate G-extension $L/\mathbb{Q}$ in which all of the primes l dividing $|G|$ split completely.

We claim that $N := LK$ is an adequate extension of K. This proves the theorem since $L/\mathbb{Q}$ and hence N/K is tamely ramified. Since $\mathfrak{p}_i^{(l)}$

has residue degree 1 over $\mathbb{Q}$, we have $K_{\mathfrak{p}_i^{(l)}} \cong \mathbb{Q}_{p_i}$ and hence:

$$(3.1) \qquad [N_{\mathfrak{P}_i^{(l)}} : K_{\mathfrak{p}_i^{(l)}}] = [L_{\mathfrak{P}_i^{(l)} \cap L} : \mathbb{Q}_{p_i^{(l)}}] \text{ for } \mathfrak{P}_i^{(l)} \,|\, \mathfrak{p}_i^{(l)}, i = 1, 2, l \,|\, |G|.$$

Letting l^{α_l} be the maximal power of l dividing $|G|$, (3.1) shows that $l^{\alpha_l} \,|\, [N : K]$ for every $l \,|\, |G|$ and hence that $\mathrm{Gal}(N/K) \cong G$. Furthermore, (3.1) shows that $D(\mathfrak{P}_i^{(l)}, N/K)$ contains an l-Sylow subgroup of G for $i = 1, 2, l \,|\, |G|$, showing that N/K is adequate, as required. □

Remark 3.4.
(1) In [14], Liedahl uses Lemma 2.5, similarly to the "only if part" of Theorem 1.3, to show that under the assumption that l does not decompose in K, the l-Sylow subgroups of a K-admissible group admit a presentation $\mathcal{M}(m, n, i, t)$ such that $\sigma_{t,n}$ fixes $K \cap \mathbb{Q}(\mu_n)$. He also uses the flexibility of [23, Theorem 1] to prove that if G itself has a presentation $\mathcal{M}(m, n, i, t)$ such that $\sigma_{t,n}$ fixes $K \cap \mathbb{Q}(\mu_n)$, then G is K-admissible.
(2) Note that the proof of the "only if part" of Theorem 1.3 applies more generally without the assumption that G is solvable.
(3) The proof of Theorem 3.3 gives furthermore that $l^{\alpha_l} \,|\, [L_{\mathfrak{P}_i^{(l)} \cap L} : \mathbb{Q}_{p_i^{(l)}}]$ for all $l \,|\, |G|, i = 1, 2$.

3.2. **Consequences.** For solvable groups G we get the following characterization of K-admissibility under the assumption that every $l \,|\, |G|$ does not decompose in K:

Corollary 3.5. *Let K be a number field and G a solvable group. Assume that every prime l that divides $|G|$ does not decompose in K. Then the following conditions are equivalent:*

(1) *There exists a tamely ramified K-adequate G-extension;*
(2) *G is K-admissible;*
(3) *for every $l||G|$, the l-Sylow subgroups of G admit a presentation $\mathcal{M}(m, n, i, t)$ such that $\sigma_{t,n}$ fixes $K \cap \mathbb{Q}(\mu_n)$.*

Proof. The implication (1) ⇒ (2) is immediate and (2) ⇒ (3) follows from Remark 3.4 ([14, Theorem 28]). The implication (3) ⇒ (1) is the "if part" of Theorem 1.3. □

Recall that a group G is called *infinitely often K-admissible* if there exist infinitely many adequate G-extensions L_i/K, $i \in \mathbb{N}$, such that $L_{r+1} \cap (L_1 \cdots L_r) = K$ (cf. [1]).

Corollary 3.6. *Let K be a number field and G a solvable group such that for every $l||G|$, the l-Sylow subgroups admit a presentation $\mathcal{M}(m, n, i, t)$ such that $\sigma_{t,n}$ fixes $K \cap \mathbb{Q}(\mu_n)$. Then G is infinitely often K-admissible.*

Furthermore, there exists a K-division algebra D that has infinitely many disjoint maximal subfields $L_i, i \in \mathbb{N}$, such that $\mathrm{Gal}(L_i/K) \cong G$.

The following lemma is useful in the proof of Theorem 3.3 and is used here to prove Corollary 3.6. Given a Galois extension $N/\mathbb{Q}$, define the following condition on a finite set of rational primes T:

(A_N) The decomposition groups $D(N/\mathbb{Q},\mathfrak{p}), p\in T, \mathfrak{p}\,|\,p$, generate $\mathrm{Gal}(N/\mathbb{Q})$.

Lemma 3.7. *Assume T satisfies (A_N). Then every finite extension $K/\mathbb{Q}$ in which the primes of T split completely is disjoint from N.*

Proof. Let $H := \mathrm{Gal}(N/N\cap K)$ and assume on the contrary that $H \neq G$. By condition (A_N) there exists a prime $p\in T$ and $\mathfrak{p}\,|\,p$ such that $D := D(N/\mathbb{Q},\mathfrak{p}) \not\subseteq H$. In particular, $[N_\mathfrak{p} : K_{\mathfrak{p}\cap K}] = |D\cap H| < |D| = [N_\mathfrak{p} : \mathbb{Q}_p]$ and hence $[K_{\mathfrak{p}\cap K} : \mathbb{Q}_p] > 1$ contradicting the assumption that p splits completely in K. □

Note that by Chebotarev's density theorem for every cyclic subgroup $C \leq G$ there are infinitely many primes $\mathfrak{p}$ of N for which $D(N/\mathbb{Q},\mathfrak{p}) = C$. Thus, we can always choose a finite set T which satisfies (A_N).

Proof of Corollary 3.6. Let S and l^{α_l} be as in the proof of Theorem 1.3. Define $D := D_0 \otimes_\mathbb{Q} K$ where D_0 is the $\mathbb{Q}$-division algebra with Hasse invariants $1/l^{\alpha_l}$ at $p_1^{(l)}$, $-1/l^{\alpha_l}$ at $p_2^{(l)}$ for $l\,|\,|G|$, and 0 at all other primes.

It suffices to show that given r disjoint G-extensions $L_1,...,L_r$ of K which are maximal subfields of D there exists a maximal subfield L_{r+1} of D such that $\mathrm{Gal}(L_{r+1}/K) = G$ and $L_{r+1}\cap(L_1\cdots L_r) = K$.

Let M be the $\mathbb{Q}$-normal closure of K and $N := L_1\cdots L_rM$. Let T be a finite set which is disjoint from S, contains all primes $l\,|\,|G|$, and for which condition (A_N) holds.

As remarked in 3.4.(3), Theorem 3.3 gives a maximal subfield L of D_0 in which the primes of T split completely. By Lemma 3.7, $L\cap N = \mathbb{Q}$ and hence $L_{r+1} := LK$ is a G-extension of K and $L_{r+1}\cap(L_1\cdots L_r) = K$. Since in addition L_{r+1} splits D, L_{r+1} is a maximal subfield of D. □

4. Proof of Theorem 3.3

In this section we prove Theorem 3.3. In Sections 4.1 and 4.2 we treat the cases of 2-groups and $\{2,3\}$-groups (groups of order 2^a3^b), respectively. We first show how the theorem follows from the latter case.

As in Section 2, we fix an embedding of an algebraic closure of $\mathbb{Q}$ into an algebraic closure of each of its completions. We shall say that a set of primes T *splits completely* in L if every prime in T splits completely in L.

Proof of Theorem 3.3. Let $n = |G|$. By [3, Lemma 1.4], there is a normal subgroup $N\lhd G$ of order prime to 2 and 3 and a $\{2,3\}$-subgroup H such that $G = NH$.

Extend T to a finite set T_0 disjoint from S which satisfies condition $(A_{\mathbb{Q}(\mu_n)})$. By the case of $\{2,3\}$-groups (Section 4.2), there exists a $\{p_1^{(2)}, p_2^{(2)}, p_1^{(3)}, p_2^{(3)}\}$-adequate H-extension $K/\mathbb{Q}$ in which $T_0 \cup \{p_1^{(l)}, p_2^{(l)}\,|\, l > 3\}$ splits completely. Since by Lemma 3.7, $K\cap\mathbb{Q}(\mu_n) = \mathbb{Q}$, and since the embedding problem $G \to \mathrm{Gal}(K/\mathbb{Q})$ splits, we may apply [17]. It follows that $K/\mathbb{Q}$ embeds into a G-extension $L/\mathbb{Q}$ such that T splits completely in L and $\mathrm{Gal}(L_{p_i^{(l)}}/\mathbb{Q}_{p_i^{(l)}})$, $i = 1, 2$, is an l-Sylow subgroup of N for all $l||N|$. In particular, $L/\mathbb{Q}$ is an S-adequate G-extension in which T splits completely, as required. □

4.1. 2**-groups.** The $\mathbb{Q}$-admissibility of metacyclic 2-group was proved in [23] using Theorem 2.1. We use Corollary 2.2 in order to prove Theorem 3.3 for 2-groups, generalizing [23]:

Proof of Theorem 3.3 for 2*-groups.* Let G be a metacyclic 2-group with presentation $G \cong \mathcal{M}(m, n, i, t)$ such that $\sigma_{t,n}$ fixes $K \cap \mathbb{Q}(\mu_n)$, and let k be the order of x in G. Let $S = \{p_1, p_2\}$ be a tame supporting set for G such that $p_i \equiv t \pmod{n}$, $i = 1, 2$.

Since S consists of odd primes, the Grunwald-Wang theorem (see Theorem 2.1.(f)) implies that there exists a $\mathbb{Z}/k$-extension $\hat{K}/\mathbb{Q}$ in which the primes of S are inert and T splits completely. We identify $\mathrm{Gal}(\hat{K}/\mathbb{Q})$ with $\langle x\rangle$ and let $K/\mathbb{Q}$ be the unique $\mathbb{Z}/m$-extension inside $\hat{K}$. The embedding problem $\pi : G \to \mathrm{Gal}(K/\mathbb{Q})$ with kernel $A := \langle y\rangle$ has a solution $\phi : G_{\mathbb{Q}} \to \langle x\rangle \subseteq G$ which is given by the restriction map to $\mathrm{Gal}(\hat{K}/\mathbb{Q})$.

Let $\pi_i : G \to \mathrm{Gal}(K_{p_i}/\mathbb{Q}_{p_i})$ be the corresponding local embedding problem at p_i, $i = 1, 2$. Since $p_i \equiv t \pmod{n}$, π_i has a surjective solution $\psi^{(i)} : G_{\mathbb{Q}_{p_i}} \to G$ whose fixed field $L^{(i)}$ is totally ramified over K_{p_i} and in particular $\mu_n \subseteq K_{p_i}$, for $i = 1, 2$ (see Section 2.2).

In order to change ϕ to a solution with the desired properties, we apply Corollary 2.3. Let $A' = \mathrm{Hom}(A, \mu_n)$ be the dual $G_{\mathbb{Q}}$-module, $K' := \mathbb{Q}(A')$, $G' = \mathrm{Gal}(K'/\mathbb{Q})$ and $G'_{p_i} := \mathrm{Gal}(K'_{p_i}/\mathbb{Q}_{p_i})$, $i = 1, 2$. Since every automorphism in $G_{\mathbb{Q}}$ that fixes A and μ_n also fixes A', we have $K' \subseteq K(\mu_n)$ and hence $K'_{p_i} \subseteq K_{p_i}(\mu_n) = K_{p_i}$, for $i = 1, 2$. Thus, G'_{p_i} is cyclic and condition 2.1.(b) holds. By Corollary 2.3, there exists a solution $\psi : G_{\mathbb{Q}} \to G$ of π, whose restriction at p_i is $\psi^{(i)}$, $i = 1, 2$, and the restriction remains the trivial solution at each $p \in T$. Since $\psi^{(1)}, \psi^{(2)}$ are surjective, ψ is also surjective. Thus, the fixed field L of $\ker\psi$ is an S-adequate G-extension of $\mathbb{Q}$ in which T splits completely, as required.

□

Remark 4.1. Note that we use Corollary 2.2 since Theorem 2.1 cannot be applied for the set $S \cup T$. In fact, the Grunwald-Wang theorem shows that the map $\rho_{S\cup T}$ need not be surjective if $2 \in T$.

4.2. $\{2, 3\}$**-groups.** Let G be a $\{2, 3\}$-group and $G(3)$ a 3-Sylow subgroup of G. If $G(3)$ is normal in G then Theorem 3.3 essentially follows from the 2-groups case by applying [17]:

Proof of Theorem 3.3 for $\{2,3\}$*-groups when* $G(3) \lhd G$.

Let $S = \{p_1^{(2)}, p_2^{(2)}, p_1^{(3)}, p_2^{(3)}\}$ be a tame supporting set for G. Extend T to a finite set T_0 disjoint from S which satisfies condition $(A_{\mathbb{Q}(\mu_n)})$ where $n = |G|$.

As shown in Section 4.1, there exists a $\{p_1^{(2)}, p_2^{(2)}\}$-adequate $G/G(3)$-extension $M/\mathbb{Q}$ in which $T_0 \cup \{p_1^{(3)}, p_2^{(3)}\}$ splits completely.

The embedding problem $G \to \mathrm{Gal}(M/\mathbb{Q})$ splits and by Lemma 3.7, $M \cap \mathbb{Q}(\mu_n) = \mathbb{Q}$. Thus, we may apply [17] and embed $M/\mathbb{Q}$ into a G-extension $L/\mathbb{Q}$ such that T splits completely in L and

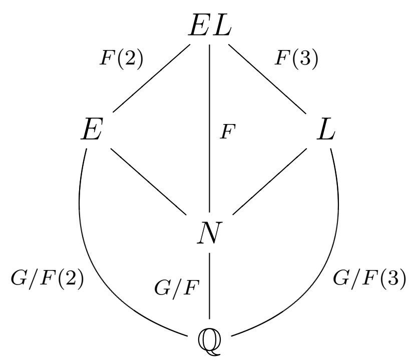
$$\mathrm{Gal}(L_{p_i^{(3)}}/\mathbb{Q}_{p_i^{(3)}}) \cong G(3) \text{ for } i = 1, 2.$$

Therefore $L/\mathbb{Q}$ is an S-adequate G-extension in which T splits completely, as required. $\square$

For $\{2,3\}$-groups G that do not have a normal 3-Sylow subgroup, we show that the proof of [24, Theorem 1] can be adjusted to give Theorem 3.3.

Let $F = F(G)$ denote the Fitting subgroup of G and $F(2)$ and $F(3)$ its 2-Sylow and 3-Sylow subgroups, respectively. The approach of [24] is to construct an adequate G/F-extension $N/\mathbb{Q}$ and embed it into an adequate $G/F(2)$-extension $E/\mathbb{Q}$ and an adequate $G/F(3)$-extension $L/\mathbb{Q}$. Since $[EL : L]$ and $[EL : E]$ are coprime, the compositum EL is an adequate G-extension of $\mathbb{Q}$.

(4.1)



When $G(3)$ is not a normal subgroup of G, [24] shows that G/F is isomorphic either to (1) S_3 or to (2) $\mathbb{Z}/3$ and that in these cases we have the following partition into subcases:

Case	G/F	$F(2)$	$G/F(3)$	2-Sylow
1.1	$\mathbb{Z}/3$	$\mathbb{Z}/2^u \times \mathbb{Z}/2^u$	$\mathbb{Z}/3 \ltimes (\mathbb{Z}/2^u \times \mathbb{Z}/2^u)$	$\mathbb{Z}/2^u \times \mathbb{Z}/2^u$
1.2	$\mathbb{Z}/3$	Q_8	$\mathrm{SL}_2(3)$	Q_8
2.1	S_3	$\mathbb{Z}/2 \times \mathbb{Z}/2$	S_4	D_8
2.2	S_3	Q_8	S_4^* or S_4^{**}	Q_{16} or D_{16}^*

Here Q_8 is the quaternions group, D_8 the dihedral group of order 8, S_4^* and S_4^{**} are the two central extensions of S_4 with kernel $\mathbb{Z}/2$, and

$$Q_{16} = \langle x, y | x^2 = y^4, y^8 = 1, x^{-1}yx = y^7\rangle,$$
$$D_{16}^* = \langle x, y | x^2 = y^8 = 1, x^{-1}yx = y^3\rangle$$

are their 2-Sylow subgroups, respectively.

In all of the above cases the 2-Sylow subgroups have unique parameters m, n and t [2].

Lemma 4.2. *Let* $G \cong \mathcal{M}(m, n, i, t)$.

(a) *If* $G \cong \mathbb{Z}/2^u \times \mathbb{Z}/2^u$ *then* $m = 2^u, n = 2^u, t = 1$.
(b) *If* $G \cong Q_8$ *then* $m = 2, n = 4, t = 3$.
(c) *If* $G \cong D_8$ *then* $m = 2, n = 4, t = 3$.
(d) *If* $G \cong D_{16}^*$ *then* $m = 2, n = 8, t = 3$.
(e) *If* $G \cong Q_{16}$ *then* $m = 2, n = 8, t = 7$.

Proof. (1) Let x, y be the generators of a presentation $\mathcal{M}(m, n, i, t)$. Since $m, n | 2^u$ and $mn = |G| = 2^{2u}$, one has $m = n = 2^u$. For $1 < t < 2^u$ the group $\mathcal{M}(2^u, 2^u, i, t)$ is non-abelian and hence $t = 1$.

(b)–(e) are conclusions from [14, Theorem 22, Case 3]. In this theorem, Liedahl gives necessary and sufficient conditions on a presentation $\mathcal{M}(m, n, i, t)$ of a group as in (b)–(e) to have an equivalent presentation with other parameters. However, these conditions require $m \geq 4$ [3] which fails for the presentations in (b)–(e). □

Proof of Theorem 3.3 for $\{2, 3\}$*-groups when* $G(3)$ *is not normal.*

We claim that the fields N, L, E in diagram (4.1) can be in fact chosen to be S-adequate extensions of $\mathbb{Q}$ in which T splits completely. This will imply that $EL/\mathbb{Q}$ is an S-adequate G-extension in which T splits completely, as required.

We first construct the field E and let $N = E^{F/(F(2))}$.

In Case (1), $G/F(2) \cong G(3)$ is of odd order and therefore [17] gives a $\{p_1^{(3)}, p_2^{(3)}\}$-adequate $G(3)$-extension $E/\mathbb{Q}$ in which $T \cup \{p_1^{(2)}, p_2^{(2)}\}$ splits completely.

In Case (2), let $q \equiv 1 (\text{mod } 8)$ be a prime which is not in $S \cup T$ and such that $p_1^{(2)} p_2^{(2)} q \equiv 1 \pmod{p}$ for all $p \in T_0 \cup \{p_1^{(3)}, p_2^{(3)}\}$. Let $k = \mathbb{Q}(\sqrt{p_1^{(2)} p_2^{(2)} q})$ and let $\mathfrak{q}$ be the prime of k which lies above q. Note that k is $\{p_1^{(2)}, p_2^{(2)}\}$-adequate and T splits completely in k. Since the embedding problem $G/F(2) \to \text{Gal}(k/\mathbb{Q})$ splits, we may apply [17] and embed k into an S-adequate $G/F(2)$-extension $E/\mathbb{Q}$ in which T and $\mathfrak{q}$ split completely. In both Cases (1) and (2), $E/\mathbb{Q}$ is S-adequate and T splits completely in E.

The construction of the field L is the same as in [24] with few modifications. Since the construction in [24] is involved and long, we do not

[2]The parameter i is also unique up to multiplication by an odd number.

[3]Note that our m is denoted as 2^m in the notation of [14].

repeat it here. A self contained version of the modified construction can be found in the author's thesis ([15]). For the reader to whom [24] is available, we give below the list of required modifications.

Note that our field N was denoted in Case (1) of [24] by k and in Case (2) by K.

(1) Replace the primes p_1, p_2 (resp. p, q) in Case (1) (resp. Case (2)) of [24] by the primes $p_1^{(2)}, p_2^{(2)}$, respectively. Since S is a supporting set, Lemma 4.2 implies that the prime $p_1^{(2)}, p_2^{(2)}$ satisfy the congruence relations required in [24] from p_1, p_2, p, q. Note that in Case (1), the primes $p_1^{(2)}, p_2^{(2)}$ split completely in $N(\mu_n)$ as required in [24]. Also note that since $p_1^{(2)}, p_2^{(2)}$ are prime to $|G|$, they are greater than 3 as required in Case (2) of [24].

(2) In Case (2), we add the prime $\mathfrak{q}$ to the modulus $\mathfrak{m}$ and require that the element γ is congruent to 1 mod $\mathfrak{q}$. The field M constructed in [24] then satisfies $\mathrm{Gal}(M_q/\mathbb{Q}_q) \cong \mathbb{Z}/2\mathbb{Z}$. Since $q \equiv 1 \pmod 8$, $\mathrm{Gal}(M_q/\mathbb{Q}_q)$ can be embedded into a $\mathbb{Z}/4\mathbb{Z}$ extension and therefore the embedding problem $G/(F(3)) \to \mathrm{Gal}(M/\mathbb{Q})$ is solvable at q as well. As shown in [24] it is solvable at all other primes and hence globally solvable.

With these changes the field L constructed in [24] gives an S-adequate $G/F(3)$-extension. In order to make the set T split completely in L, we make the following additional modifications:

(1) In Case (1.1), the embedding problem $G/(F(3)) \to \mathrm{Gal}(N/\mathbb{Q})$ splits and hence has the trivial solution ϕ. Instead of applying Theorem 2.1, we apply Corollary 2.3 insuring the same prescribed conditions at S but in addition that the solution remains trivial at primes of T.

(2) In Cases (1.2) and (2), we add the primes of N that lie over primes of T to the modulus $\mathfrak{m}$ and require that $\gamma \equiv 1 \pmod{\mathfrak{p}}$ for every $\mathfrak{p} \cap \mathbb{Q} \in T$. This insures that T splits completely in K (resp. in M) in Case (1.2) (resp. Case (2)).

Let ϕ be the solution obtained in Case (1.2) (resp. Case (2)) of [24] to the embedding problem $G/F(3) \to \mathrm{Gal}(K/\mathbb{Q})$ (resp. $G/F(3) \to \mathrm{Gal}(M/\mathbb{Q})$). We apply Theorem 2.1 in order to change ϕ to a solution ψ which is trivial at primes of T. Since the local embedding problem at $p_i^{(2)}$ is Frattini, ψ is surjective at $p_i^{(2)}$, $i = 1, 2$. Thus, the fixed field L of $\ker \psi$ is S-adequate and T splits completely in L.

□

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Department of Mathematics, 530 Church St., University of Michigan, Ann Arbor 48109.
E-mail address: neftin@umich.edu