Correction to Lemma 1.8 in "Patching and admissibility over two-dimensional complete local domains"

We are grateful to Yong HU for pointing out to us a gap in the proof of [1, Lemma 1.8], namely the isomorphism $R/\mathfrak{p} \cong S/\mathfrak{q}$ implies only $S = R + \mathfrak{q}$ and not $S = R + \mathfrak{p}S$ as required for this argument.

Lemma 1.8 in [1] is applied for the rings

$$R_0 = D_{J\cup J'}, R_1 = D_J, R_2 = D_{J'}, R = D_\emptyset,$$

where $J \cap J' = \emptyset$, to show that $S := R_1 \cap R_2 = R$. Let us show this assertion directly. In particular, this will trivially imply that for these rings $\mathfrak{q} = \mathfrak{p}S$.

Recall that I is a finite set and that v is the extension of the order function of the ideal $\mathfrak{p} := (x, y) \lhd K[x, y]$ to $K(x, y)$. For $i \in I$, let $z_i = \frac{y}{x-c_iy}$ and for a subset $J \subset I$, D_J is defined as the completion of $K[z_j|j \in J][x, y]$ with respect to v.

Lemma 0.1. *Let $i, j \in I$ be two distinct indices. Then $D_{\{i\}} \cap D_{\{j\}} = D_\emptyset$.*

Proof. By [1, Proposition 1.5], $D_{\{i\}} = K[z_i][[x-c_iy]]$ and hence an element $f \in D_{\{i\}}$ can be written as $\sum_{k=0}^\infty f_k(z_i)(x - c_iy)^k$ for some polynomials f_k, $k \geq 0$. Assume $f \in D_{\{i,j\}}$ can also be written as $\sum_{k=0}^\infty g_k(z_j)(x - c_jy)^k \in D_{\{j\}} = K[z_j][[x - c_jy]]$, where g_k are polynomials for $k \geq 0$. In particular, one has

$$f_k(z_i)(x - c_iy)^k = g_k(z_j)(x - c_jy)^k \pmod{\mathfrak{p}^{k+1}D_{\{i,j\}}} \tag{0.1}$$

for all $k \geq 0$. We claim that equality (0.1) in fact holds in $D_{\{i,j\}}$. Indeed since $x - c_iy = (1 + (c_j - c_i)z_j)(x - c_jy)$, the difference between the LHS and RHS in (0.1) is:

$$(f_k(z_i)(1 + (c_j - c_i)z_j)^k - g_k(z_j))(x - c_jy)^k \in k[z_i, z_j](x - c_jy)^k. \tag{0.2}$$

Since $\mathfrak{p}$ is contained in the center, [1, Proposition 1.5] implies that the difference (0.2) is in $\mathfrak{p}^{k+1}D_{\{i,j\}}$ only if it is zero, proving the claim.

By finding a common denominator, one can write an element $f_k(z_i)(x - c_iy)^k$ as $\frac{p_k(x,y)}{(x-c_iy)^m}$ where $m \geq 0$ and p_k is a homogenous polynomial of degree $k + m$ that is prime to $(x - c_iy)^m$. Writing $g_k(z_j)(x - c_jy)^k = \frac{q_k(x,y)}{(x-c_jy)^l}$ for $l \geq 0$ and q_k a homogenous polynomial of degree $k + l$ that is prime to $(x - c_iy)^l$, the equality $\frac{p_k(x,y)}{(x-c_iy)^m} = \frac{q_k(x,y)}{(x-c_jy)^l}$ implies that $m = l = 0$ and hence that $f_k(z_i)(x-c_iy)^k \in K[x, y]$ for all $k \geq 0$. It follows that $f = \sum_{k=0}^\infty f_k(z_i)(x - c_iy)^k \in K[[x, y]]$ as required. □

Let us complete the proof of Proposition 1.10 in [1]:

Proposition 0.2. *Suppose $J, J' \subseteq I$. Then $D_J \cap D_{J'} = D_{J\cap J'}$.*

Proof. Clearly $D_{J\cap J'} \subseteq D_J \cap D_{J'}$. For the converse inclusion, we distinguish between two cases. First suppose that $J \cap J' \neq \emptyset$ and fix $j \in J \cap J'$. Then $D_J = K[z_k|k \in J][[x - c_jy]]$, $D_{J'} = K[z_k|k \in J'][[x - c_jy]]$ and hence

$$D_J \cap D_{J'} = (K[z_k|k \in J] \cap K[z_k|k \in J'])[[x - c_jy]].$$

By [1, Lemma 1.9], $K[z_k|k \in J] \cap K[z_k|k \in J'] = K[z_k|k \in J \cap J']$.

Now suppose that $J \cap J' = \emptyset$. If $|J| = |J'| = 1$, then the claim follows from Lemma 0.1. Assume without loss of generality $|J| \geq 2$. For $j_1, j_2 \in J$, we have

$D_J \cap D_{J' \cup \{j_i\}} = D_{\{j_i\}}$, for $i = 1, 2$. In particular $D_J \cap D_{J'} \subseteq D_{\{j_1\}} \cap D_{\{j_2\}}$. By Lemma 0.1, $D_{\{j_1\}} \cap D_{\{j_2\}} = D_\emptyset$ implying that $D_J \cap D_{J'} = D_\emptyset$ as required. □

References

[1] D. Neftin and E. Paran, Patching and admissibility over complete local domains of dimension 2, *Algebra and Number theory*, Vol. 4 (2010), No. 6, 743-762.