FIELDS OF DEFINITION FOR ADMISSIBLE GROUPS

DANNY NEFTIN AND UZI VISHNE

Abstract. A finite group G is admissible over a field M if there exists a division algebra whose center is M, which has a maximal subfield which is a Galois extension of M, with Galois group G.

We consider the interaction of this property over M with a subfield $K \subset M$, comparing variations where the division algebra, the maximal subfield or the Galois group are asserted to be defined over K. We completely determine the logical implications between those variants.

1. Introduction

A group G is admissible over a field M if there exists a division algebra D whose center is M, that has a maximal subfield L which is a Galois extension of M with Galois group G. Such a division algebra is called a G-crossed product. The notion of admissibility was introduced and studied by Schacher [17], where a necessary and sufficient criterion for the admissibility of G over a number field M was given in terms of an inverse Galois problem with local conditions (see Theorem 3.1). Subsequently, the criterion was used to determine the admissibility of various families of groups, e.g. [4], [20], [2], [3], [9].

There is an intricate connection between admissibility over M and admissibility over subfields K of M. Given a crossed product division algebra D_0 over K, the algebra $D = D_0 \otimes_K M$ is a crossed product with respect to the same group, and is often also a division algebra. This simple procedure completely explains M-admissibility in several recent works, with fields such as $K = \mathbb{C}((x))(t)$ in [5], $K = \mathbb{C}((x, y))$ in [10], and $K = \mathbb{Q}_p(x)$ for a certain family S of groups in [15, §3]. In these cases, for a finite extension M/K and every M-admissible group G (with $G \in S$ if $K = \mathbb{Q}_p(x)$), there exists a G-crossed product division algebra D_0 over K whose restriction to M is a division algebra.

It therefore makes sense to ask, for a given extension M/K and an M-admissible group G, how well can admissibility be realized over K. When a crossed product division algebra D over M is obtained by restriction of scalars of a crossed product D_0 over M with respect to the same group, we say that the crossed product D is defined over K. This is a strong condition which implies the other variations studied in this paper. Failing this strong assumption, it is still possible that G is both K and M-admissible; that the G-crossed product D is defined over K (namely, $D = D_0 \otimes_K M$ for a suitable division algebra over K); that L is defined

Key words and phrases. Admissible group; adequate field; admissibility over a subfield; tame admissibility.

Vishne was partially supported by the U.S.-Israel Binational Science Foundation (grant no. 2010/149).

and is Galois over K (namely, $L = L_0 \otimes_K M$ where L_0/K is G-Galois); or that L is merely defined over K.

This paper studies eight variations of M-admissibility of a group G, with respect to a fixed subfield K of M. These are defined in Section 2, where we also provide a complete diagram of implications between the conditions. In Section 5 we provide counterexamples to every implication we do not prove, with G being a p-group and M a number field.

We show that over number fields M/K, if a cyclic group G is M-admissible, then a suitable crossed product is realizable over K (see Section 3). For other groups, much weaker variations of this claim fail.

In Section 4 we consider tame admissibility, which is the type of admissibility that is best understood over number fields (see e.g. [11]) and show that these eight variants hold with respect to tame admissibility. The difference between tame and wild admissibility is an essential ingredient in the construction of counterexamples in Section 5. A condition of different flavor, involving extensions of M which are Galois over K, is presented and compared to the eight variations in Section 6.

2. Conditions on the field of definition and the main theorem

2.1. **The eight variations.** We present eight variations on admissibility over M with respect to a subfield M. Let K be a field and G a finite group. We shall say that a field L is K-*adequate* if it is a maximal subfield in some division algebra whose center is K. We shall say that L is a G-*extension* of K if L/K is a Galois extension with Galois group $\mathrm{Gal}(L/K) \cong G$.

Let M/K be a finite field extension. One way to study the condition

(1) G is M-admissible

is by refining it to require that the crossed-product division algebra or its maximal subfield are defined over K (we say that a field or an algebra over M is defined over K if it is obtained by scalar extension from K to M).

Condition (1) requires the existence of an M-adequate G-extension L/M. Two ways in which L can be related to K provide the following variants:

(2) there exists an M-adequate G-extension L/M for which L is defined over K; *or*
(3) there exists an M-adequate G-extension L/M so that $L = L_0 \otimes M$ for some field L_0 for which $\mathrm{Gal}(L_0/K) \cong G$.

For the algebra D to be defined over K, we may require that:

(4) there exists a K-division algebra D_0 and a G-extension L/M for which L is a maximal subfield of $D_0 \otimes M$; *or*
(5) there exists a K-division algebra D_0 and a maximal subfield L_0 which is a G-extension of K so that $L_0 \cap M = K$ and $L = L_0M$ is a maximal subfield of the division algebra $D = D_0 \otimes M$.

If $L = L_0 \otimes_K M$, the interaction between L_0 and L may involve the division algebras:

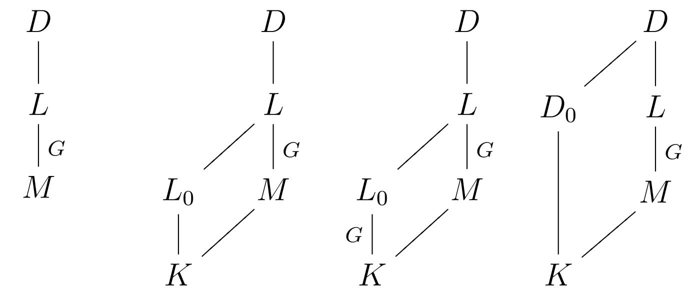
(6) there exists a K-adequate G-extension L_0/K for which L_0M is an M-adequate G-extension; *or*

(7) there exists a K-adequate G-extension L_0/K for which L_0M is an M-adequate G-extension.

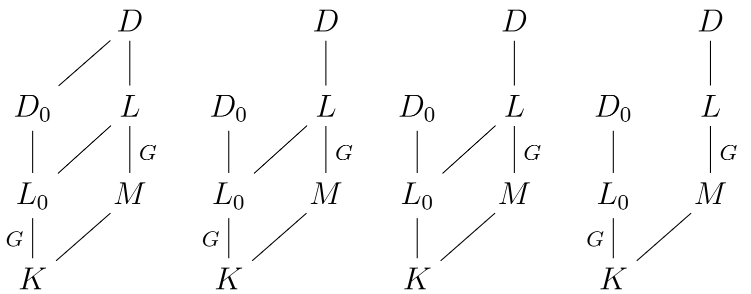
And finally we have the double condition

(8) G is both K-admissible and M-admissible.

We provide a diagrammatic description of each condition, for easy reference. Inclusion is denoted by a vertical line, and diagonal lines show the extension of scalars from K to M. A vertical line is decorated by G if the field extension is G-Galois. Note that in some cases ((3), (5) and (6)) the fact that the extension L_0/K is Galois implies that L/M is Galois as well, with the same Galois group.



(1) (2) (3) (4)



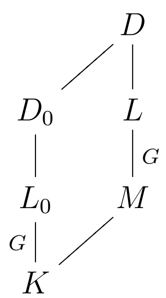
(5) (6) (7) (8)

We shall say that a triple (K, M, G) satisfies Condition (m) if there are L_0, L, D_0 and D as required in this condition. In such case we shall also say (L_0, L, D_0, D) realizes Condition (m), omitting L_0 or D_0 if they are not needed.

Remark 2.1. Let M/K be a finite extension of fields and G a finite group. One might also consider the condition

($5'$) there exists a G-crossed product K-division algebra D_0, for which $D = D_0 \otimes M$ is also a G-crossed product division algebra.

Here there is no explicit assumption that the maximal subfields be related; in the spirit of previous diagrams, this condition is described by



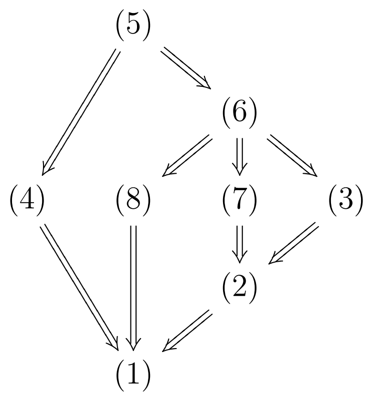
However, ($5'$) is equivalent to Condition (5). Indeed, suppose that (L_0, L, D_0, D) realizes ($5'$). Then D is of index $|G|$ and D is also split by $L' = ML_0$. Therefore $[L' : M] = |G|$, $L_0 \cap M = K$ and hence we can take L' to be the required maximal G-subfield of D. Thus, (L_0, L', D_0, D) realizes (5). The converse implication is obvious, taking $L = L_0 \otimes_K M \subset D_0 \otimes_K M = D$.

2.2. **The logical implications.** The following theorem describes the relation between the eight variants:

Theorem 2.2. *Of the eight conditions in Subsection 2.1,* (m) *implies* (n) *for all finite extensions* M/K *and finite groups* G*, if and only if the diagram of* (n) *can be obtained from the diagram of* (m) *by removal of lines and decorations.*

More explicitly, the implications in Diagram 2.1 always hold, and every other implication fails for some extension of number fields and some finite p*-group:*

(2.1)



We prove the implications given in diagram 2.1 here. The counterexamples for every other implication are given in Section 5.

Proof of positive part of Theorem 2.2. Fix K, M and G. Clearly if (L_0, L, D_0, D) realizes Condition (5), (L_0, L, D_0, D) also realizes (6) and (L, D_0, D) realizes (4), so that $(5) \Rightarrow (4), (6)$.

If (L_0, L, D_0, D) realizes (6) then L_0/K is a G-extension and hence $L = L_0M/M$ is also a G-extension (since $L_0 \cap M = K$). Thus, (L_0, L, D_0, D) realizes (7). It is clear that L_0 is a field of definition of L (and $\mathrm{Gal}(L_0/K) = G$) and hence (L_0, L, D) realizes (3). As L_0 is a K-adequate G-extension and L is an M-adequate G-extension, (L_0, L, D_0, D) realizes (8). Therefore $(6) \Rightarrow (3), (8), (7)$.

If (L_0, L, D) realizes Condition (3) then $\mathrm{Gal}(L_0/K) = G$, $\mathrm{Gal}(L/M) = G$ (since $L_0 \cap M = K$) and hence (L_0, L, D_0, D) realizes Condition (2). If (L_0, L, D_0, D) realizes condition (7), clearly L_0 is a field of definition of L and hence (L_0, L, D) realizes Condition (2).

Clearly when (K, M, G) satisfies either of the conditions $(2), (4), (8)$, G is M-admissible and hence $(2), (4), (8) \Rightarrow (1)$. □

In Section 5 we provide counterexamples to all the implications which were not proved here, thus showing that the diagram in Theorem 2.2 depicts the precise logical interaction between the eight conditions.

3. Cyclic groups over number fields

For a prime v of a number field K, we denote by K_v the completion of K with respect to v. If L/K is a finite Galois extension, L_v denotes the completion of L with respect to some prime divisor of v in L.

The basic criterion for admissibility over number fields is due to Schacher:

Theorem 3.1 ([17]). *Let K be a number field and G a finite group. Then G is K-admissible if and only if there exists a Galois G-extension L/K such that for every rational prime p dividing $|G|$, there is a pair of primes v_1, v_2 of K such that each of $\mathrm{Gal}(L_{v_i}/K_{v_i})$ contains a p-Sylow subgroup of G.*

We use this criterion in the construction of counterexamples in Section 5 and to prove the following proposition:

Proposition 3.2. *Let G be a cyclic group. Then Conditions (1)–(8) are satisfied for every extension of number fields M/K.*

Proof. It is sufficient to show that (5) is satisfied. By Chebutarev density Theorem (applied to the Galois closure of M/K) there are infinitely many primes v of K that split completely in M. Let v_1, v_2 be two such primes that are not divisors of 2. By the weak version (prescribing degrees and not local extensions) of the Grunwald-Wang Theorem (see [21, Corollary 2] or [1, Chapter 10]) there exists a G-extension L_0/K for which $\mathrm{Gal}((L_0)_{v_i}/K_{v_i}) = G$ and thus L_0 is K-adequate, so there is a division algebra D_0 containing L_0 as a maximal subfield, and supported by $\{v_1, v_2\}$. As v_i split completely in M we have $L = L_0M$ satisfies $\mathrm{Gal}(L_{v_i}/M_{v_i}) = G$ for $i = 1, 2$ and hence $\mathrm{Gal}(L/M) = G$. Finally $D = D_0 \otimes M$ is a division algebra by the choice of the v_i. Thus L is M-adequate and (K, M, G) satisfies (5). □

We mention in this context the 'linear disjointness' (LD) of number fields, as defined and established in [14, Prop. 2.7]: for every finite extension M/K in characteristic 0, every central simple algebra over K contains a maximal separable subfield P that is linearly disjoint from M over K. This notion can be bypassed by appealing to the Chebutarev density, as above.

4. Tame admissibility

The conditions of Section 2 can also be considered with respect to tame K-admissibility. Let us recall the definition of tame admissibility.

For an extension of fields L/K, $\mathrm{Br}(L/K)$ denotes the kernel of the restriction map $\mathrm{res}: \mathrm{Br}(K) \to \mathrm{Br}(L)$. Let $\mathrm{Br}(L/K)_{\mathrm{tr}}$ be the subgroup of the relative Brauer group $\mathrm{Br}(L/K)$ that consists of the Brauer classes which are split by the maximal tame subextension of L_v/K_v, for every prime v of L.

Over a number field K, the exponent of a division algebra is equal to its index, and so L is K-adequate if and only if there exists an element of order $[L:K]$ in $\mathrm{Br}(L/K)$ ([17, Proposition 2.1]). Following this observation one defines:

Definition 4.1. Let K be a number field. We say that a finite extension L of K is *tamely K-adequate* if there exists an element of order $[L:K]$ in $\mathrm{Br}(L/K)_{\mathrm{tr}}$.

Likewise, a finite group G is *tamely K-admissible* if there exists a tamely K-adequate G-extension L/K.

4.1. **Liedahl's condition.** Let μ_n denote the set of n-th roots of unity in $\mathbb{C}$. For t prime to n, let $\sigma_{t,n}$ be the automorphism of $\mathbb{Q}(\mu_n)/\mathbb{Q}$ defined by $\sigma_{t,n}(\zeta) = \zeta^t$ for $\zeta \in \mu_n$.

Definition 4.2. We say that a metacyclic p-group G satisfies *Liedahl's condition* (first defined in [7]) with respect to K, if it has a presentation

$$G = \left\langle x, y \mid x^m = y^i,\ y^n = 1,\ x^{-1}yx = y^t \right\rangle \tag{4.1}$$

such that $\sigma_{t,n}$ fixes $K \cap \mathbb{Q}(\mu_n)$.

It follows from [7] (see also [8, Corollary 2.1.7]) that tamely K-admissible groups G have metacyclic p-Sylow subgroups that satisfy Liedahl's condition for every prime divisor p of $|G|$. There are no known counterexamples to the opposite implication. In fact if a metacyclic p-group satisfies Liedahl's condition over K then it is realizable over infinitely many completion of K (see [7]).

Remark 4.3. Note that if a metacyclic p-group G satisfies Liedahl's condition over M, then it satisfies the condition over every subfield K.

The following is shown in [7, Theorem 30] for G a p-group, and in [8, Theorem 2.3.1] for G solvable.

Theorem 4.4. *Let K be a number field and G a solvable group whose Sylow subgroups satisfy Liedahl's condition. Then G is tamely K-admissible.*

Remark 4.5. In fact the proof of [8, Theorem 2.3.1] shows that there exists a G-extension L_0/K and $D_0 \in \mathrm{Br}(L_0/K)_{tr}$ such that $D := D_0 \otimes_{\mathbb{Q}} K$ remains a division algebra.

In particular $L := L_0 \otimes_K M$ is an M-adequate field which is a G-extension of M. Thus, not only G is M-admissible but there exists also a G-crossed product division algebra D and a maximal subfield L so that both are defined compatibly over $\mathbb{Q}$.

As a corollary one has (see [8]):

Corollary 4.6. *Let K be a number field. Let G be a solvable group such that the rational prime divisors of $|G|$ do not decompose (i.e. have a unique prime divisor) in K. Then G is K-admissible if and only if its Sylow subgroups are metacyclic and satisfy Liedhal's condition.*

4.2. **Fields of definition for tame admissibility.** The conditions of Section 2 can also be considered with respect to tame K-admissibility. Let G be a solvable group and K, M number fields. By Proposition 4.4, if G is tamely M-admissible then there exists a tamely K-adequate G-extension L_0/K for which $L = L_0M$ is M-adequate (and hence tamely M-adequate). For $m = 1, \ldots, 8$, let (m^*) denote the condition (m), where every adequate extension is assumed to be tamely adequate, and an admissible group is assumed tamely admissible. More precisely for $m = 4, 5$ we consider

(4^*) there exists a K-division algebra D_0 and a G-extension L/M for which $[D] = [D_0 \otimes M] \in \mathrm{Br}(L/M)_{\mathrm{tr}}$ and L is a maximal subfield of D,

and

(5*) there exists a K-division algebra D_0 and a maximal subfield L_0 which is a G-extension of K so that $L_0 \cap M = K$, $D_0 \in \mathrm{Br}(L_0/K)_{\mathrm{tr}}$ and $L = L_0M$ is a maximal subfield of $D = D_0 \otimes M$ (and hence $[D] \in \mathrm{Br}(L/M)_{\mathrm{tr}}$).

Corollary 4.7. *Let G be a solvable group and M/K a finite extension of number fields. Then the conditions* (1^*)–(8^*) *are all equivalent.*

Proof. With the added conditions the implications given in (2.1) clearly continue to hold. But by Remark 4.5 the implication $(1^*) \Rightarrow (5^*)$ also holds. □

5. Examples

In this section we give counterexamples for all the implications not claimed in Theorem 2.2. In all the examples, the group G is a p-group. This shows that Diagram 2.1 describes all the correct implications even for p-groups.

Let us first show that neither one of the conditions (4) or (8) imply any other condition except (1). For this, by the implication Diagram 2.1, it is sufficient to show that $(4) \not\Rightarrow (8)$, $(8) \not\Rightarrow (4)$, $(4) \not\Rightarrow (2)$ and that $(8) \not\Rightarrow (2)$. We will show that $(8) \not\Rightarrow (4)$ by demonstrating that $(6) \not\Rightarrow (4)$. In fact an example for $(6) \not\Rightarrow (4)$ will show that no other condition, except (5), implies Condition (4). To complete the proof we should also prove $(7) \not\Rightarrow (8)$, $(7) \not\Rightarrow (3)$, $(3) \not\Rightarrow (8)$ and $(3) \not\Rightarrow (7)$.

The first example is relies on:

Remark 5.1. If F_1 and F_2 are field extensions of F such that $L = F_1 \otimes_F F_2$ is a field, and F_1/F and L/F_1 are Galois, then L is Galois over F.

Example 5.2 ($(4) \not\Rightarrow (2), (8) \not\Rightarrow (2)$). Let $p \equiv 1 \pmod 4$, $G = (\mathbb{Z}/p\mathbb{Z})^3$ and $K = \mathbb{Q}(i, \sqrt{p})$. Note that p splits in K. Denote the prime divisors of p in K by v_1, v_2.

Let $\overline{K_{v_i}(p)}^{\mathrm{ab}}$ be the maximal abelian pro-p extension of K_{v_i}. By local class field theory the Galois group $\mathrm{Gal}(\overline{K_{v_i}(p)}^{\mathrm{ab}}/K_{v_i})$ is isomorphic to the pro-p completion of the group $K_{v_i}^*$ which is $\mathbb{Z}_p^n$ where $n = [K_{v_i} : \mathbb{Q}_p] + 1 = 3$ (see [19], Chapter 14, Section 6).

Since $K_{v_1} = K_{v_2} = \mathbb{Q}_p(\sqrt{p})$ this shows G is realizable over K_{v_1}, K_{v_2}. By the Grunwald-Wang Theorem there a $(\mathbb{Z}/p^2\mathbb{Z})^3$-extension $\hat{M}/K$ such that $\hat{M}_{v_i}$ is the maximal abelian extension of exponent p^2 of K_{v_i}, namely the unique $(\mathbb{Z}/p^2\mathbb{Z})^3$-extension of K_{v_i}. Let $M = \hat{M}^G$, so that $\mathrm{Gal}(M/K) \cong G$.

Since $\hat{M}/M$ and M/K both have full local degrees at v_1, v_2, both are adequate G-extensions. By choosing $L = \hat{M}$ and $L_0 = M$, we deduce that (K, M, G) satisfies condition (8). To show that (K, M, G) satisfies (4) it suffices to notice that v_1, v_2 have unique prime divisors w_1, w_2 in M. Every division algebra D whose invariants are supported in $\{w_1, w_2\}$ is K-uniformly distributed and hence $D \in \mathrm{Im}(\mathrm{res}_K^M)$. Take D with

$$\mathrm{inv}_{w_1}(D) = \frac{1}{p^3}, \quad \mathrm{inv}_{w_2}(D) = -\frac{1}{p^3}$$

and $\mathrm{inv}_w(D) = 0$ for every other prime w of M. We then have $D \in \mathrm{Im}(\mathrm{res}_K^M)$, D is a G-crossed product division algebra and hence (K, M, G) satisfies (4).

Let us show (2) is not satisfied. Suppose on the contrary that there exists a triple (L_0, L, D) realizing (2). By Remark 5.1, L/K is Galois and

$$\mathrm{Gal}(L/K) \cong \mathrm{Gal}(L/L_0) \ltimes \mathrm{Gal}(L/M) \cong G \ltimes_\phi G$$

via some homomorphism $\phi : G \to \mathrm{Aut}(G) = \mathrm{GL}_3(\mathbb{F}_p)$. As G is a p-group, ϕ is a homomorphism into some p-Sylow subgroup P of $\mathrm{GL}_3(\mathbb{F}_p)$. These are all conjugate, so we can choose a basis $\{v_1, v_2, v_3\}$ of $\mathbb{F}_p^3$ for which P is the Heisenberg group (in other words the unipotent radical of the standard Borel subgroup), generated by the transformations:

$$\phi_x(a,b,c) = (a+b,b,c),\ \phi_y(a,b,c) = (a,b+c,c),\ \phi_u(a,b,c) = (a+c,b,c)$$

which correspond to the matrices

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, u = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that P has the presentation

$$P = \langle x, y, u \mid x^p = y^p = u^p = [x,u] = [y,u] = 1, [y,x] = u\rangle.$$

Every subgroup of the form $\mathbb{F}_p^2 \ltimes G$ is a maximal subgroup of $G \ltimes G$ and thus the Frattini subgroup Φ of $G \ltimes_\phi G$ is contained in $1 \ltimes G$. Now the subgroup $H = \langle v_1, v_2\rangle \le G$ is invariant under the action of P and hence under the action of G via ϕ. So, $G \ltimes_\phi H \le G \ltimes_\phi G$ is a maximal subgroup and $\Phi \le 1 \ltimes H$. This shows that $\dim_{\mathbb{F}_p} G/\Phi \ge 4$ and thus $G \ltimes_\phi G$ is not generated by less than 4 elements. Therefore $G \ltimes G$ is not realizable over $\mathbb{Q}_p(\sqrt{p})$.

On the other hand both L/M and M/K have full rank at w_i and v_i and hence $\mathrm{Gal}(L_{w_i}/K_{v_i}) = G \ltimes G$ which is a contradiction as $G \ltimes G$ is not realizable over K_{v_i}. Thus, (K, M, G) does not satisfy Condition (2).

Example 5.3 ((7) $\not\Rightarrow$ (8), (7) $\not\Rightarrow$ (3)). Let $p \equiv 1 \pmod 4$, $K = \mathbb{Q}(i)$ and v_1, v_2 the two prime divisors of p in K. Let $G = \mathbb{F}_p^p$ and $P = \mathbb{F}_p \wr (\mathbb{Z}/p\mathbb{Z})$ so that $P = G \rtimes \langle x\rangle$ where $x^p = 1$.

The maximal p-extension $\overline{\mathbb{Q}_p(p)}$ has Galois group $\overline{G_{\mathbb{Q}_p}(p)} := \mathrm{Gal}(\overline{\mathbb{Q}_p(p)}/\mathbb{Q}_p)$ which is a free pro-p group on two generators. As P is generated by two elements it is realizable over $\mathbb{Q}_p$. Since P is a wreath product of abelian groups it has a generic extension over K and hence by [16] there exists a P-extension L/K for which $\mathrm{Gal}(L_{v_i}/K_{v_i}) = P$ for $i = 1, 2$. Let us choose $M = L^G$ the G-fixed subfield of L.

Then clearly L/M is an M-adequate extension which is defined over K since

$$\mathrm{Gal}(L/K) \cong \mathrm{Gal}(M/K) \ltimes \mathrm{Gal}(L/M).$$

The subfield $L_0 = L^{\langle x\rangle}$ is K-adequate since $[(L_0)_{v_i} : K_{v_i}] = p^p$ for $i = 1, 2$ and hence (K, M, G) satisfies Condition (7).

(We write $(L_0)_{v_i}$ even though L_0/K is not Galois, since v_i has a unique prime divisor in L_0 for $i = 1, 2$.)

Now since G is an abelian group of rank $p > 2$, G is not realizable over $K_{v_1}, K_{v_2} \cong \mathbb{Q}_p$ and hence not K-admissible. It follows that (K, M, G) does not satisfy Condition (8). In order for (K, M, G) to satisfy Condition (3) there should be a G-extension L_0/K for which L_0M is M-adequate. In particular,

$\mathrm{Gal}((L_0M)_{v_1}/M_{v_1}) \cong G$ and hence $\mathrm{Gal}((L_0)_{v_1}/K_{v_1}) \cong G$ which contradicts the fact that G is not realizable over $K_{v_1} \cong \mathbb{Q}_p$. Thus (K, M, G) does not satisfy Condition (3) either.

Example 5.4 ((3) $\not\Rightarrow$ (8), (3) $\not\Rightarrow$ (7)). Let $p \equiv 1 \pmod 4$ and v be its unique prime divisor in $K = \mathbb{Q}(\sqrt{p})$. Let $M = \mathbb{Q}(\sqrt{p}, i)$ and $G = (\mathbb{Z}/p\mathbb{Z})^3$.

By the Grunwald-Wang Theorem, there exists a Galois G-extension L_0/K for which $\mathrm{Gal}((L_0)_v/K_v) = G$. Thus $L = L_0M$ is a Galois G-extension of M such that $\mathrm{Gal}(L_{v_i}/M_{v_i}) = G$ for each of the two prime divisors v_1, v_2 of v in M. It follows that L is M-adequate and (K, M, G) satisfies Condition (3). But as p has a unique prime divisor in K and G is not metacyclic, G is not K-admissible and hence (K, M, G) does not satisfy Condition (8).

Let us also show that (K, M, G) does not satisfy Condition (7). Assume, on the contrary, that (L_0, L, D_0, D) realizes (7). Then, as L_0 is K-adequate there are two primes w_1, w_2 of K for which $[(L_0)_{w_i} : K_{w_i}] = |G|$. Without loss of generality we assume $w_1 \neq v$ (otherwise take w_2). Then $\mathrm{Gal}(L_{w_1}/M_{w_1}) \cong G$ since $(L_0)_{w_1} \cap M_{w_1} = K_{w_1}$. This is a contradiction since tamely ramified extensions (such as L_{w_1}/M_{w_1}) have metacyclic Galois groups. Thus (K, M, G) does not satisfy Condition (7).

Remark 5.5. Let us also show that (K, M, G) does not satisfy (4) (so that this example will also show that (3) $\not\Rightarrow$ (4)). Assume on the contrary that there exsits a tuple (L, D_0, D) that realizes (4). Since D contains L as a maximal subfield, $\mathrm{Gal}(L_{v_i}/M_{v_i}) = G$ and $\mathrm{inv}_{v_i}(D) = \frac{m_i}{p^3}$ where $(m_i, p) = 1$, for $i = 1, 2$. Note that G is realizable over M_v only for divisors v of p, so that $\mathrm{inv}_u(D) = \frac{m_u}{p^2}$ for suitable $m_u \in \mathbb{Z}$ for every $u \neq v_1, v_2$. Now, since D is in the image of the restriction, we have $m_1 = m_2$. The sum of M-invariants of D is an integer and hence $p \mid m_1 + m_2 = 2m_1$ which contradicts $(m_i, p) = 1$.

Example 5.6 ((4) $\not\Rightarrow$ (8)). Let p be an odd prime, and q a prime $\equiv 1 \pmod p$. Let $K = \mathbb{Q}(\sqrt{p})$, so that q splits (completely) in K. Let v be the prime divisor of p in K and w a prime divisor of q in K. Let M be a $\mathbb{Z}/p\mathbb{Z}$-extension of K in which v splits completely and w is inert. Let $G = (\mathbb{Z}/p\mathbb{Z})^3$.

Consider the K-division algebra D_0 whose invariants are:

$$\mathrm{inv}_v(D_0) = \frac{1}{p^3}, \quad \mathrm{inv}_w(D_0) = -\frac{1}{p^3}$$

and $\mathrm{inv}_u(D_0) = 0$ for every other prime u of K. Now $D = D_0 \otimes_K M$ has M-invariants $\mathrm{inv}_{v_i}(D) = \frac{1}{p^3}$ for the prime divisors $v_1, v_2, \ldots, v_p$ of v in M, $\mathrm{inv}_{w'}(D) = -\frac{1}{p^2}$ for the prime divisor w' of w and $\mathrm{inv}_u(D) = 0$ for every other prime u of M. Note that G is realizable over $M_{v_i} \cong K_v$ and since $q \equiv 1 \pmod p$, $(\mathbb{Z}/p\mathbb{Z})^2$ is realizable over $M_{w'}$. By the Grunwald-Wang Theorem, there exists a Galois G-extension L/M for which:

$$\mathrm{Gal}(L_{v_i}/M_{v_i}) = G \text{ for } i = 1, \ldots, p, \text{ and } \mathrm{Gal}(L_{w'}/M_{w'}) = (\mathbb{Z}/p\mathbb{Z})^2.$$

Thus L is a maximal subfield of D and (K, M, G) satisfies condition (4). Since p has a unique prime divisor in K and G is not metacyclic we deduce G is not K-admissible and hence (K, M, G) does not satisfy Condition (8).

Example 5.7 ((6) $\not\Rightarrow$ (4)). Let $p \geq 13$ be a prime such that $p \equiv 1 \pmod 4$. Let $K = \mathbb{Q}(\mu_p)$ and $M = \mathbb{Q}(\mu_{4p^2}) = \mathbb{Q}(i, \mu_{p^2})$. Let G be the following metacyclic group of order p^3:

$$G = \left\langle x, y \mid x^p = y^{p^2} = 1,\ x^{-1}yx = y^{p+1} \right\rangle. \tag{5.1}$$

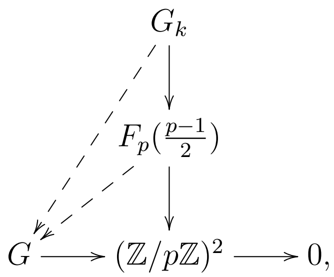
Note that p splits in $\mathbb{Q}(i)$ and has exactly two prime divisors v_1, v_2 in M. Let u be the unique prime divisor of p in K.

Let us first show that (K, M, G) does not satisfy Condition (4). As M does not satisfy Liedahl's condition, G is not realizable over M_v for every $v \neq v_1, v_2$. Assume on the contrary there exists an M-adequate G-extension L/M and an M-division algebra D which is defined over K and has a maximal subfield L. Then necessarily: $\mathrm{inv}_{v_1}(D) = \mathrm{inv}_{v_2}(D) = \frac{a}{p^3}$ for some $(a, p) = 1$. But as the sum of invariants of D is 0 and G is not realizable over any other v we have $p \mid 2a$ or $p \mid a$. Contradiction.

To prove that (K, M, G) satisfies Condition (6) we shall need the following lemma:

Lemma 5.8. *Let $p \geq 11$ be a prime, $k = \mathbb{Q}_p(\mu_p)$ and G the group defined in* (5.1)*. Then, given a G-extension m/k, there exists a G-extension l/k for which $m \cap l = k$.*

Proof. For every G-extension l/k we note that $\mathrm{Gal}(l \cap m/k)$ is an epimorphic image of G and as such it is either G or an abelian group. Thus if l intersects with m non-trivially then it also intersects with $m' = m^{\langle y^p \rangle}$ (the fixed field of y^p which also corresponds to the abelianization of G). We note that $\mathrm{Gal}(m'/k) = (\mathbb{Z}/p\mathbb{Z}) \times (\mathbb{Z}/p\mathbb{Z})$. The maximal abelian group realizable over k is of rank $p-1$, and since $\frac{p-1}{2} \geq 4$ there exists a $(\mathbb{Z}/p\mathbb{Z})^2$-extension l'/k which is disjoint from m' and for which the epimorphism $\pi : G_k \to \mathrm{Gal}(l'/k)$ splits through a free pro-p group of rank $\frac{p-1}{2}$. Thus l' is disjoint from m' and hence to m. Embedding l' into a G-extension produces a G-extension which is disjoint to m. This is possible since the following embedding problem for G_k:



splits through a free pro-p group of large enough rank and hence has a surjective solution. ∎

Let us prove Condition (6) is satisfied. Let $\sigma_{p+1} \in \mathrm{Gal}(\mathbb{Q}(\mu_{p^2})/\mathbb{Q})$ be the automorphism that sends $\sigma_{p+1}(\zeta) = \zeta^{p+1}$ where ζ is a primitive root of unity of order p^2. Thus σ_{p+1} fixes μ_p and hence $\sigma_{p+1} \in \mathrm{Gal}(\mathbb{Q}(\mu_{p^2})/K)$. As G satisfies Liedahl's condition over K, G is realizable over infinitely many primes of K (see the proof of [7, Theorem 29] or [8, Theorem 2.3.1]), so choose one such prime w which is not a divisor of p. Since $[K_u : \mathbb{Q}_p] = p - 1 \geq 11$, it follows from Lemma 5.8 that G is also realizable over K_u and furthermore there exists a G-extension L_0^p/K_u for which $M_u \cap L_0^p = K_u$.

By Theorems 6.4(b) and 2.5 of [13] (see also [8, Proposition 1.2.13]), there exists a G-extension L_0/K for which $\mathrm{Gal}((L_0)_w/K_w) = G$ and $(L_0)_u = L_0^p$. Hence L_0 is K-adequate. Let $L = L_0M$. As $M_u \cap L_0^p = K_u$ we have $\mathrm{Gal}(L_{v_i}/M_{v_i}) = G$ for $i = 1, 2$. Thus L/M is an M-adequate G-extension and (K, M, G) satisfies Condition (6). This concludes the proof of Example 5.7.

6. Galois extensions of the subfield

Having established the connections between Conditions (1)–(8), we consider in this section a final condition where the maximal subfield of the division algebra is Galois over K:

(9) there exists an M-adequate G-extension L/M for which L is Galois over K;

Proposition 6.1. *The implication (9)⇒(1) holds. On the other hand (9)*$\not\Rightarrow$ (m) *for* $m = 2, \ldots, 8$, *and* $(m) \not\Rightarrow (9)$ *for* $m = 1, \ldots, 8$.

Before providing a proof we start with some remarks. Proposition 3.2 shows that when G is cyclic, Conditions (1)–(8) are satisfied for every extension of number fields M/K. This is not the case for (9):

Example 6.2. If M/K is not normal, (9) does not necessarily hold for a cyclic group G. Let $n \geq 2$ and M/K be an extension of degree n whose Galois closure M' has Galois group $\mathrm{Gal}(M'/K) = S_n$. Then every field $L \supseteq M$, which is Galois over K, must contain M' and hence there is no (adequate) C_n-extension L/M for which L/K is Galois.

Remark 6.3. Remark 5.1 shows that if M/K is Galois then (2) ⇒ (9). In particular (9) holds for G cyclic if M/K is Galois.

Proof. Clearly when (K, M, G) satisfies (9), G is M-admissible and hence (1) holds. Example 6.2 shows that (5) $\not\Rightarrow$ (9), and hence no other condition among (1)- (8) implies (9). The examples presented in Section 5 show that (9) does not imply any of the conditions (2), (8), (4). Indeed:

- (9) $\not\Rightarrow$ (2) follows from Example 5.2 where (K, M, G) does not satisfy (2), but $L = \hat{M}$ was constructed to be Galois over K and hence (9) holds.
- (9) $\not\Rightarrow$ (8) follows from Example 5.3 where (8) does not hold, but (7) does. Since by Remark 6.3, (7) ⇒ (2) ⇒ (9) as M/K is Galois, (K, M, G) also satisfies (9).
- (9) $\not\Rightarrow$ (4): Consider Example 5.7. By Remark 6.3, as M/K is Galois, (6) ⇒ (9). Thus, (K, M, G) in this example also satisfies (9).

This completes the placement of (9) in Diagram 2.1 of Theorem 2.2. □

References

[1] E. Artin, J. Tate, Class field theory. Advanced Book Classics. Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA, 1990.
[2] W. Feit, The K-admissibility of $2A6$ and $2A7$. Israel J. Math. 82 (1993), 141–156.
[3] W. Feit, $\mathrm{SL}(2, 11)$ is $\mathbb{Q}$-admissible. J. Algebra 257 (2002), 244–248.
[4] B. Gordo, M. Schacher, The admissibility of A_5. J. Number Theory 11 (1979), 498–504.
[5] D. Habater, J. Hartmann, D. Krashen, Patching subfields of division algebras. Trans. AMS. 363 (2011), 3335–3349.
[6] J. P. Labute, Classification of Demushkin groups. Canad. J. Math. 19 (1967), 106–132.

[7] S. Liedahl, Presentations of metacylic p-groups with applications to K-admissibility questions. J. Algebra 169 (1994), 965–983.
[8] D. Neftin, Admissibility of finite groups over number fields. Ph.D. Thesis, Technion 2011.
[9] D. Neftin, Tamely ramified subfields of division algebras. J. Algebra 378 (2013), 184–195.
[10] D. Neftin, E. Paran, Patching and admissibility over two-dimenisonal complete local domains. Algebra and Number Theory 4 (2010), 743–762.
[11] D. Neftin, U. Vishne, Realizability and admissibility under extension of p-adic and number fields. Doc. Math. 18 (2013) 359–382.
[12] J. Neukirch, On solvable number fields. Invent. Math. 53 (1979), 135–164.
[13] J. Neukirch, Uber das Einbettungsproblem der algebraischen Zahlentheorie. Invent. Math. 21 (1973), 59–116.
[14] A.S. Rapinchuk and I.R. Rapinchuk, On division algebras having the samemaximal subfields. Manuscripta Math. 132 (2010), 273–293.
[15] B. S. Reddy, V. Suresh, Admissibility of groups over function fields of p-adic curves. Adv. Math. 237 (2013), 316–330.
[16] D. Saltman, Generic Galois extensions. Proc. Nat. Acad. Sci. U.S.A. 77 (1980), 1250–1251.
[17] M. Schacher, Subfields of division rings. I. J. Algebra 9 (1968), 451–477.
[18] J.-P. Serre, Galois Cohomology. Springer, 1964 (English trans. 1996).
[19] J.-P. Serre, Local fields. Graduate Texts in Mathematics 67. Springer-Verlag, New York-Berlin, 1979.
[20] J. Sonn, Q-admissibility of solvable groups. J. Algebra 84 (1983), 411–419.
[21] S. Wang, On Grunwald's theorem. Ann. of Math. (2) 51 (1950), 471–484.

Deptartment of Mathematics, Technion-Israel Institute of Technology, Haifa 32000, Israel
E-mail address: `dneftin@tx.technion.ac.il`

Deptartment of Mathematics, Bar Ilan University, Ramat Gan 52900, Israel
E-mail address: `vishne@math.biu.ac.il`