GALOIS SUBFIELDS OF TAME DIVISION ALGEBRAS

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Abstract. We show that a finite dimensional tame division algebra over a Henselian field F has a maximal subfield Galois over F if and only if the residue division algebra $\overline{D}$ has a maximal subfield Galois over the residue field $\overline{F}$. This gives a criterion for such an algebra to be a crossed product, leads to the discovery of new noncrossed products and, if $\overline{F}$ is a global field, to a description of their location among tame division algebras.

1. Introduction

We consider finite-dimensional division algebras D over their center F. The division algebra D is called a *crossed product* if it contains a maximal subfield which is Galois over F, otherwise a *noncrossed product.* The question of existence of noncrossed products arose in the 1930's, and was answered affirmatively by Amitsur in 1972 [2]. Subsequently, their existence and location in the Brauer group for more familiar fields F was studied using a variety of methods.

Supposing that F is a Henselian valued field, we recall that the valuation of F extends uniquely to a valuation of D for every division algebra D over F. In case D is inertially split, i.e. split by an unramified extension of F, it was first known that D is a crossed product only if the residue division algebra $\overline{D}$ is a crossed product [10, Thm. 5.15(b)]. This goes back to Saltman [13] who used this kind of argument to construct new noncrossed products from old. By a more complete criterion [7], any inertially split D is a crossed product if and only if $\overline{D}$ contains a maximal subfield Galois over the residue field $\overline{F}$. Note that this is a stronger condition than saying $\overline{D}$ is a crossed product since $\overline{F}$ may be a proper subfield of the center of $\overline{D}$. This criterion goes back to Brussel [3], who used this kind of argument over complete rank 1 valued fields to obtain noncrossed products over fields as simple as $\mathbb{Q}((t))$. Subsequently, the criterion has led to a description of the "location" of crossed and noncrossed products among all inertially split division algebras over F, when F is Henselian with global residue field, [6], [5].

In this paper we consider the larger class of division algebras D which are *tamely ramified (tame)* over a Henselian field F, i.e. split by a tamely ramified extension of F. In particular, these include all division algebras whose degree is prime to

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the residue characteristic. Our main result, Theorem 1.1, generalizes the above mentioned criterion to tame division algebras.

Theorem 1.1. *Let F be a Henselian field, and D a finite-dimensional tamely ramified division algebra with center F. Then D has a maximal subfield Galois over F if and only if $\overline{D}$ has a maximal subfield Galois over $\overline{F}$.*

Theorem 1.1 is useful in determining which tame division algebras are crossed products over all Henselian fields F whose residue field is sufficiently well understood. When $\overline{F}$ is of cohomological dimension 1 or a local field, we deduce that all tame division algebras over F are crossed products. When $\overline{F}$ is a global field we describe the location of noncrossed products among tame division algebras, extending [6] and [5], and discovering new noncrossed products.

The main difficulty in proving Theorem 1.1 lies in the "only if" implication. Given a maximal subfield M of D, Galois over F, neither the residue field $\overline{M}$ itself nor the compositum of $\overline{M}$ with the center of $\overline{D}$ need to be a maximal subfield of $\overline{D}$. Hence in the tame case the construction is significantly different from the inertially split case, where it was enough to consider $\overline{M} \cdot Z(\overline{D})$, cf. [7].

Our construction of the desired maximal subfield of $\overline{D}$ uses the theory of graded division algebras which provides a one to one correspondence between tame division algebras and graded division algebras [9]. It associates to D a graded division algebra $\mathsf{gr}(D)$ over the associated graded field $\mathsf{gr}(F)$, and to M a graded maximal subfield $\mathsf{gr}(M)$ Galois over $\mathsf{gr}(F)$. Most importantly, it equips $\mathsf{gr}(D)$ with canonical subalgebras which we use in order to modify $\mathsf{gr}(M)$ to a maximal graded subfield M' of $\mathsf{gr}(D)$ with residue field which is maximal in $\overline{D}$ and Galois over $\overline{F}$. We note that the canonical subalgebras are available only in graded setting, and hence an analogues argument seems impossible to carry on without this setting.

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2. Graded and valued algebras

We first recall the basic definition and facts concerning graded algebras, based on [9] with the exception of Section 2.3 which is based on [8] and [11]. A more extensive treatment of these facts will appear in [16].

Throughout the section we let Γ be *a torsion free abelian group*.

2.1. **Graded rings and division algebras.** A *graded ring* D *with grade group* Γ (or a Γ-*graded ring*) is a ring with a direct sum decomposition $\mathsf{D} = \oplus_{\gamma\in\Gamma}\mathsf{D}_\gamma$, where each D_γ is an abelian group and $\mathsf{D}_\gamma \cdot \mathsf{D}_\delta \subseteq \mathsf{D}_{\gamma+\delta}$. Denote $\Gamma_\mathsf{D} = \{\gamma \in \Gamma \mid \mathsf{D}_\gamma \neq \{0\}\}$. A *graded homomorphism* $\phi : \mathsf{D} \to \mathsf{E}$ of Γ-graded rings is a homomorphism which preserves the grading, i.e. $\phi(\mathsf{D}_\gamma) \subseteq \mathsf{E}_\gamma$ for all $\gamma \in \Gamma$.

A *graded subring* of D is a subring $\mathsf{E} \subseteq \mathsf{D}$ such that $\mathsf{E} = \oplus_{\gamma\in\Gamma_D}(\mathsf{D}_\gamma \cap \mathsf{E})$. This decomposition defines a Γ-grading on E with $\mathsf{E}_\gamma = \mathsf{D}_\gamma \cap E$. Note that the centralizer

$C_{\mathsf{D}}(\mathsf{E})$ is a graded subalgebra of D, hence in particular the center $Z(\mathsf{D})$ is graded subalgebra of D.

Let D be a graded ring with $1 \neq 0$. Then D is called a *graded division ring* if every nonzero element of D_γ, $\gamma \in \Gamma$, is a unit. In this case D_0 is a division ring, and multiplication by γ induces an isomorphism $\mathsf{D}_\gamma \cong \mathsf{D}_0$ of D_0-module; Hence D_γ is a rank 1-module over D_0.

Commutative graded division rings are called *graded fields*. If D is a graded division ring whose center contains a graded field F as a graded subring, then D is called a *graded division algebra* over F. In this case D_0 is a division algebra over F_0. For a graded F-subalgebra $\mathsf{E} \subseteq \mathsf{D}$, D is free as an E-module, cf. [9, Paragraph preceeding (1.6)], and the dimension $[\mathsf{D} : \mathsf{E}]$ is defined as the rank of D as an E-module. We shall assume that *all graded division algebras are finite-dimensional.* The *degree* of a graded division algebra D is $\deg \mathsf{D} := \sqrt{[\mathsf{D} : Z(\mathsf{D})]}$.

Given two Γ-graded algebras D and E over F, the tensor product $\mathsf{D} \otimes_{\mathsf{F}} \mathsf{E}$ is also a Γ-graded algebra with $(\mathsf{D} \otimes_{\mathsf{F}} \mathsf{E})_\gamma$ generated by $d_\alpha \otimes e_\beta$ where $d_\alpha \in D_\alpha, e_\beta \in E_\beta$, and $\alpha + \beta = \gamma$. The double centralizer theorem is available in the graded setting [9, Proposition 1.5] and implies, using the same argument as in the ungraded setting, that for a graded division algebra D over F, $[\mathsf{M} : \mathsf{F}] \leq \deg \mathsf{D}$ for any graded subfield M, with equality if and only if M is a maximal graded subfield of D.

2.2. **Ramification.** The following ramification properties of graded division rings are analogues to ramification properties of valued division rings. Let D be a graded division ring, and $\mathsf{E} \subseteq \mathsf{D}$ a graded division subring. Then one easily obtains the fundamental equality, cf. [9, (1.7),p.79],

$$[\mathsf{D} : \mathsf{E}] = [\mathsf{D}_0 : \mathsf{E}_0]|\Gamma_{\mathsf{D}} : \Gamma_{\mathsf{E}}|. \tag{2.1}$$

We call D *unramified* over E if $\Gamma_{\mathsf{D}} = \Gamma_{\mathsf{E}}$ (i.e. $[\mathsf{D} : \mathsf{E}] = [\mathsf{D}_0 : \mathsf{E}_0]$) and *totally ramified* if $[\mathsf{D} : \mathsf{E}] = |\Gamma_D : \Gamma_E|$ (i.e. $\mathsf{D}_0 = \mathsf{E}_0$).

Let $\mathsf{A} \subseteq \mathsf{D}$ be a graded subring which is also a graded division ring and contains E. Then D/E is unramified (resp. totally ramified) if and only if D/A and A/E are both unramified (resp. totally ramified).

Let F be a graded subfield of $Z(\mathsf{D})$, so that D is a graded F-division algebra. For every F_0-subalgebra A of D_0 there is a unique graded division F-subalgebra $\mathsf{A} \subseteq \mathsf{D}$ with $\mathsf{A}_0 = A$. This A is generated by A over F and is canonically isomorphic to $A \otimes_{\mathsf{F}_0} \mathsf{F}$.

Note that the intersection $\mathsf{A} \cap \mathsf{B}$ of two graded subrings of D is a graded subring with $(\mathsf{A} \cap \mathsf{B})_\gamma = \mathsf{A}_\gamma \cap \mathsf{B}_\gamma$. In the totally ramified case we have:

Lemma 2.1. *Let* A, B *be two graded subrings of* D *which are also graded division rings. Assume* D *is totally ramified over* B. *Then*

(i) $\mathsf{B} = \oplus_{\gamma \in \Gamma_B} \mathsf{D}_\gamma$*;*
(ii) $\Gamma_{A \cap B} = \Gamma_{\mathsf{A}} \cap \Gamma_{\mathsf{B}}$.

Proof. Clearly, $\mathsf{B} \subseteq \mathsf{B}' := \oplus_{\gamma\in\Gamma_B} D_\gamma$. Since D_γ is a rank-1 module over $\mathsf{D}_0 = \mathsf{B}_0$, one has $\mathsf{B}_\gamma = \mathsf{D}_\gamma$, for all $\gamma \in \Gamma_\mathsf{B}$. Hence, $\mathsf{B}' = \mathsf{B}$, showing (i).

The inclusion $\Gamma_{\mathsf{A}\cap\mathsf{B}} \subseteq \Gamma_\mathsf{A} \cap \Gamma_\mathsf{B}$ is obvious. Conversely, for $\gamma \in \Gamma_\mathsf{A} \cap \Gamma_\mathsf{B}$, using $\mathsf{B}_\gamma = \mathsf{D}_\gamma$, we have $(\mathsf{A}\cap\mathsf{B})_\gamma = \mathsf{A}_\gamma \cap \mathsf{D}_\gamma = \mathsf{A}_\gamma \neq \{0\}$, completing (ii). □

2.3. **Graded field extensions.** Let L/F be a finite extension of Γ-graded fields. As Γ is torsion free, F is a domain and we can form its field of quotients $q(\mathsf{F})$. Note that $[\mathsf{L} : \mathsf{F}] = [q(\mathsf{L}) : q(\mathsf{F})]$, [8, Proposition 2.1].

We call L *tame* (or *tamely ramified*) over F if the field extension $\mathsf{L}_0/\mathsf{F}_0$ is separable and $\operatorname{char} \mathsf{F}_0 \nmid |\Gamma_\mathsf{L} : \Gamma_\mathsf{F}|$. Normal and Galois extensions of graded fields can be defined similarly to the ungraded case, and are equivalent to the corresponding properties of their quotient fields:

(2.2)

L/F	$q(\mathsf{L})/q(\mathsf{F})$	Reference
tame	separable	[8, Proposition 3.5]
normal	normal	[11, Lemma 1.2]
Galois	Galois	[11, Proposition 1.3]

Furthermore, the group of graded automorphisms $Aut(\mathsf{L}/\mathsf{F})$ is naturally isomorphic to $Aut(q(\mathsf{L})/q(\mathsf{F}))$ [8, Corollary 2.5(d)], and if L/F is Galois there is a one to one correspondence between graded subfields $\mathsf{F} \subseteq \mathsf{M} \subseteq \mathsf{L}$ and subfields $q(\mathsf{F}) \subseteq M \subseteq q(\mathsf{L})$ which preserves degrees and Galois groups [8, Proposition 5.1]. In particular, if L/F is normal and $\mathsf{F} \subseteq \mathsf{M} \subseteq \mathsf{L}$ is a graded subfield then:

(2.3) M/F is normal if and only if M is invariant under every $\sigma \in Aut(\mathsf{M}/\mathsf{F})$.

Consider commuting graded subfields K, L, of a graded division algebra D which contain $\mathsf{F} := Z(\mathsf{D})$. The *compositum* $\mathsf{K}\cdot\mathsf{L}$ is the algebra generated by K and L in D. Picking F_0-bases $\{k_\gamma\}_{\gamma\in\Gamma_\mathsf{K}}$ and $\{l_\delta\}_{\delta\in\Gamma_\mathsf{L}}$ for K and L, respectively, which consist of homogenous elements, one has

$$\mathsf{K}\cdot\mathsf{L} = \sum_{\gamma\in\Gamma_\mathsf{K},\delta\in\Gamma_\mathsf{L}} \mathsf{F}_0 k_\gamma l_\delta,$$

and hence $\mathsf{K}\cdot\mathsf{L}$ is a graded subfield of D. We say that K and L are *linearly disjoint* over $\mathsf{E} := \mathsf{K}\cap\mathsf{L}$ if the natural map $\mathsf{K}\otimes_\mathsf{E}\mathsf{L} \to \mathsf{K}\cdot\mathsf{L}$ is an isomorphism. In particular, K and L are linearly disjoint over E if and only if $[\mathsf{K}\cdot\mathsf{L} : \mathsf{E}] = [\mathsf{K} : \mathsf{E}][\mathsf{L} : \mathsf{E}]$.

Lemma 2.2. *Let D be a graded division algebra over F. Let K and L be commuting graded subfields of D containing F, with L Galois over $\mathsf{K}\cap\mathsf{L}$. Then K and L are linearly disjoint over $\mathsf{K}\cap\mathsf{L}$.*

Proof. Let $\mathsf{E} := \mathsf{K}\cap\mathsf{L}$. By [8, Corollary 2.5(a)], K (resp. L) is the integral closure of E in $q(\mathsf{K})$ (resp. $q(\mathsf{L})$). Hence $q(\mathsf{K})\cap q(\mathsf{L}) = q(\mathsf{E})$. It is straightforward to verify that $q(\mathsf{K})\cdot q(\mathsf{L}) = q(\mathsf{K}\cdot\mathsf{L})$. As $q(\mathsf{L})/q(\mathsf{E})$ is Galois, $q(\mathsf{K})$ and $q(\mathsf{L})$ are linear disjointness over $q(\mathsf{E})$ and hence $[\mathsf{K}\cdot\mathsf{L} : \mathsf{K}] = [q(\mathsf{K})\cdot q(\mathsf{L}) : q(\mathsf{K})] = [q(\mathsf{L}) : q(\mathsf{E})] = [\mathsf{L} : \mathsf{E}]$. □

We shall also need the following lemma concerning totally ramified extensions:

Lemma 2.3. *Let* $\mathsf{F} \subseteq \mathsf{K} \subseteq \mathsf{M}$ *be graded fields such that* M/K *is totally ramified. If* M/F *is normal (resp. Galois) then* K/F *is normal (resp. Galois).*

Proof. Since M/K is totally ramified, by Lemma 2.1 we have $\mathsf{K} = \bigoplus_{\gamma \in \Gamma_{\mathsf{K}}} \mathsf{M}_\gamma$. As each $\sigma \in \mathrm{Gal}(\mathsf{M}/\mathsf{F})$ is degree-preserving, this shows that $\sigma(\mathsf{K}) = \mathsf{K}$. If M is normal over F then by (2.3), K is also normal over F. If M/F is tame then by (2.2), K/F is tame. Both statements are proved. □

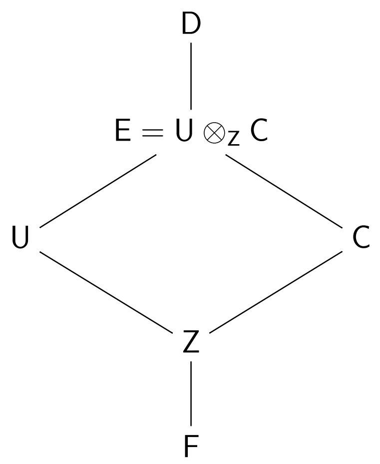
Finally, we also note that the equivalent definition of normality in [11] and [8, Corollary 2.4(c)] imply:

(2.4) if L/F is normal (resp. Galois) then $\mathsf{L}_0/\mathsf{F}_0$ is normal.

2.4. **Canonical subalgebras of graded division algebras.** The following canonical algebras and their properties were introduced in [9]. Let D be a graded division algebra over its center F.

Then D has the following canonical subalgebras:

$$\begin{aligned} \mathsf{U} &= \mathsf{D}_0 \otimes_{\mathsf{F}_0} \mathsf{F} = \text{the maximal subalgebra of } \mathsf{D} \text{ unramified over } \mathsf{F}, \\ \mathsf{Z} &= Z(\mathsf{D}_0) \otimes_{\mathsf{F}_0} \mathsf{F} = \text{the center of } \mathsf{U}, \\ \mathsf{C} &= \text{the centralizer } C_{\mathsf{D}}(\mathsf{U}), \\ \mathsf{E} &= \text{the centralizer } C_{\mathsf{D}}(\mathsf{Z}) = \mathsf{U} \otimes_{\mathsf{Z}} \mathsf{C}. \end{aligned}$$



Note that U and hence Z, C, and E, were chosen canonically and hence are invariant under conjugation by nonzero elements of D_γ, $\gamma \in \Gamma_{\mathsf{D}}$. Thus, the same holds for Z, E, and C. Moreover, as D is totally ramified over U, one has $\mathsf{U}_0 = \mathsf{E}_0 = \mathsf{D}_0$. In particular,

$$[\mathsf{D} : \mathsf{E}] = |\Gamma_{\mathsf{D}} : \Gamma_{\mathsf{E}}| = [\mathsf{Z} : \mathsf{F}] = [\mathsf{Z}_0 : \mathsf{F}_0]. \tag{2.5}$$

The center $\mathsf{Z}_0 = Z(\mathsf{D}_0)$ clearly contains F_0 but is not necessarily equal to it. In fact, $\mathsf{Z}_0/\mathsf{F}_0$ is Galois with abelian Galois group which is described as follows. Let $int(d_\gamma)$ denote the inner automorphism which sends $x \in \mathsf{D}$ to $d_\gamma x d_\gamma^{-1}$. Let $\theta_{\mathsf{D}} : \Gamma_{\mathsf{D}} \to \mathrm{Gal}(\mathsf{Z}_0/\mathsf{F}_0)$ be the homomorphism for which $\theta_{\mathsf{D}}(\gamma)$ is the restriction of

$int(d_\gamma)$ to Z_0 for any nonzero $d_\gamma \in \mathsf{D}_\gamma$. Then θ_{D} is well defined, surjective, and its kernel is Γ_E, [9, Proposition 2.3]. Hence $\mathrm{Gal}(\mathsf{Z}_0/\mathsf{F}_0) \cong \Gamma_{\mathsf{D}}/\Gamma_{\mathsf{E}}$.

We shall need the following properties of maximal graded subfields of C:

Lemma 2.4. *Let* T *be a maximal graded subfield of* C. *Then:*

(i) T *is Galois over* F*;*

(ii) $\Gamma_{\mathsf{T}} = \Gamma_{C_{\mathsf{E}}(\mathsf{T})}$

Proof. By the double centralizer theorem [9, Proposition 1.5],

$$C_{\mathsf{E}}(\mathsf{T}) = C_{\mathsf{U}}(\mathsf{Z}) \otimes_{\mathsf{Z}} C_{\mathsf{C}}(\mathsf{T}) = \mathsf{U} \otimes_{\mathsf{Z}} \mathsf{T}.$$

As $\Gamma_{\mathsf{U}} = \Gamma_{\mathsf{Z}} \subseteq \Gamma_{\mathsf{T}}$, we get $\Gamma_{C_{\mathsf{E}}(\mathsf{T})} = \Gamma_{\mathsf{T}}$, proving (ii).

To show (i), we first claim that T/Z is Galois. Since C is totally ramified over Z, its graded subfield T is also totally ramified over Z. Hence, by Lemma 2.1

$$\mathsf{T} = \bigoplus_{\gamma \in \Gamma_{\mathsf{T}}} \mathsf{C}_\gamma. \tag{2.6}$$

By [9, Proposition 2.3], we have $\mathrm{char}\, \mathsf{F}_0 \nmid |\Gamma_{\mathsf{C}} : \Gamma_{\mathsf{Z}}|$, and hence $\mathrm{char}\, \mathsf{F}_0 \nmid |\Gamma_{\mathsf{T}} : \Gamma_{\mathsf{Z}}|$. This shows that T is tame over Z. As T is also totally ramified over Z, and $\mathrm{char}\, \mathsf{F}_0 \nmid |\Gamma_T : \Gamma_Z|$, [8, Proposition 3.3] implies that $q(\mathsf{T})/q(\mathsf{Z})$ is a Kummer extension, hence normal. As T/Z is tame and normal it is Galois, proving the claim.

As Z is Galois and unramified over F, we have $\mathrm{Gal}(\mathsf{Z}/\mathsf{F}) \cong \mathrm{Gal}(\mathsf{Z}_0/\mathsf{F}_0)$. Let $\sigma \in \mathrm{Gal}(\mathsf{Z}/\mathsf{F})$. Since θ_{D} is surjective, there is a unit $d \in \mathsf{D}_\gamma$ for some $\gamma \in \Gamma$, such that $int(d)|_{\mathsf{Z}_0} = \sigma|_{Z_0}$, and hence $int(d)|_{\mathsf{Z}} = \sigma$. Since d lies in one of the components D_γ, $int(d)$ preserves C. Thus, (2.6) shows that $int(d)$ preserves T. Since Z/F and T/Z are Galois and since every automorphism in $\mathrm{Gal}(\mathsf{Z}/\mathsf{F})$ extends to a graded automorphism of T, it follows that T is Galois over F, as required. □

2.5. **Correspondence with valued division algebras.** Let D be a division algebra over a Henselian field F, v the unique extension of the valuation on F to D, and Γ its value group (a totally ordered abelian group). Recall that D is tame if and only if

$$[D : F] = [\overline{D} : \overline{F}][\Gamma_D : \Gamma_F],$$

$Z(\overline{D})$ is separable over $\overline{F}$, and $\mathrm{char}\, \overline{F} \nmid [\ker(\theta_D) : \Gamma_F]$, cf. [10, §6]. Furthermore, D is *inertial* if it is tame and unramified, i.e. $\Gamma_D = \Gamma_F$.

Tame division algebras D correspond to graded division algebras by associating to D a Γ-graded division ring $\mathsf{gr}(D)$ whose components are $\mathsf{gr}(D)_\gamma = D_{\geq\gamma}/D_{>\gamma}$, where

$$D_{\geq\gamma} = \{x \in D \mid v(x) \geq \gamma\} \text{ and } D_{>\gamma} = \{x \in D \mid v(x) > \gamma\}.$$

Furthermore, $\mathsf{gr}(D)$ is a graded division algebra over $\mathsf{gr}(F)$ with $\Gamma_{\mathsf{D}} = \Gamma_D$. The map $D \to \mathsf{gr}(D)$ gives a degree preserving bijection [9, Theorem 5.1] between tamely ramified division algebras over F (up to isomorphism) and graded division algebras over $\mathsf{gr}(F)$ (up to isomorphism). By [9, Corollary 5.7], this correspondence is functorial under field extensions L/F, and hence L is a maximal subfield of D

if and only if $\mathsf{gr}(L)$ is a maximal graded subfield of $\mathsf{gr}(D)$. By [11, Theorem 1.5], if L/F is normal then so is $\mathsf{gr}(L)/\mathsf{gr}(F)$.

On the level of fields, by [8, Theorem 5.2], there is a correspondence between tame graded extensions of $\mathsf{gr}(F)$ and tame field extensions of F, which preserves degrees and Galois groups. In particular, every tame graded field extension L of $\mathsf{gr}(F)$ corresponds to a tame field extension L/F, called a tame lift, such that $\mathsf{gr}(L) \cong \mathsf{L}$ as graded fields.

We shall also need the following lemma which describes properties that are preserved under tensor products with inertial algebras. Denote by $[D] \in \mathrm{Br}(F)$ the Brauer class of D, let $\deg[D] = \deg D := \sqrt{[D:F]}$, and $\theta_D := \theta_{\mathsf{gr}(D)}$ [1].

Lemma 2.5. *(Jacob-Wadsworth* [10]*) Let I, D be central division algebras over F. Assume I is inertial and D is tame. Let D' be the division algebra underlying $[I \otimes_F D]$. Then $Z(\overline{D'}) \cong Z(\overline{D})$, $\theta_D = \theta_{D'}$, $[\overline{D'}] = [\overline{I} \otimes_{\overline{F}} \overline{D}]$ in $\mathrm{Br}(Z(\overline{D}))$, and the following ratio is preserved:*

$$\frac{\deg D}{\deg \overline{D}} = \frac{\deg D'}{\deg \overline{D'}}. \tag{2.7}$$

Proof. All assertions are proved in [10, Corollary 6.8], except for the last which is derived as follows. Since D is tame:

$$\begin{aligned}[D:F] &= [\mathsf{gr}(D):\mathsf{gr}(F)] = [\overline{D}:\overline{F}]\cdot[\Gamma_D:\Gamma_F] \\ &= [\overline{D}:Z(\overline{D})][Z(\overline{D}):\overline{F}][\ker\theta_D:\Gamma_F]|\operatorname{Im}\theta_D|.\end{aligned}$$

Since $[Z(\overline{D}):\overline{F}] = |\operatorname{Im}\theta_D|$, by taking square roots, we get:

$$\deg D = \deg \overline{D}\cdot|\operatorname{Im}\theta_D|\cdot\sqrt{[\ker\theta_D:\Gamma_F]}. \tag{2.8}$$

Thus,

$$\frac{\deg D}{\deg \overline{D}} = |\operatorname{Im}\theta_D|\cdot\sqrt{[\ker\theta_D:\Gamma_F]} = |\operatorname{Im}\theta_{D'}|\cdot\sqrt{[\ker\theta_{D'}:\Gamma_F]} = \frac{\deg D'}{\deg \overline{D'}}. \qquad \square$$

3. Maximal subfields of tame graded division algebras

Throughout the section we fix a graded division algebra D with center F, and let $\mathsf{Z}, \mathsf{C}, \mathsf{U}$, and E be its canonical subalgebras (introduced in §2.4). We first prove the graded version of Theorem 1.1:

Theorem 3.1. *A finite-dimensional graded division algebra D has a graded maximal subfield Galois (resp. normal) over F if and only if D_0 has a maximal subfield Galois (resp. normal) over F_0.*

The following Proposition gives the "if" implication of Theorem 3.1.

[1] This definition slightly differs from the definition of θ_D in [10], where θ_D is defined on Γ_D/Γ_F.

Proposition 3.2. *Let M be a maximal subfield of D_0, $\mathsf{L} = M \otimes_{\mathsf{F}_0} \mathsf{F}$, and T a maximal graded subfield of C. Then $\mathsf{M} := \mathsf{L} \cdot \mathsf{T}$ is a maximal graded subfield of D. Furthermore, if M/F_0 is Galois (resp. normal) then M/F is Galois (resp. normal).*

Proof. Since M is maximal it contains $Z(\mathsf{D}_0)$. By definition of $\mathsf{Z}, \mathsf{U}, \mathsf{C}$, cf. §2.4, we have $\mathsf{Z} \subseteq \mathsf{L} \subseteq \mathsf{U}$, and L and T commute. Hence, M is a graded subfield of D. Since L/Z is inertial we have:

$$[\mathsf{L}{:}\mathsf{Z}] \;=\; [M{:}Z(\mathsf{D}_0)] \;=\; \deg \mathsf{D}_0 \;=\; \deg \mathsf{U}.$$

As L/Z is unramified, and T/Z is totally ramified one has $\mathsf{L} \cap \mathsf{T} = \mathsf{Z}$. By Lemma 2.4, T/Z is Galois. Hence, Lemma 2.2 implies:

$$[\mathsf{M}{:}\mathsf{Z}] \;=\; [\mathsf{L}{:}\mathsf{Z}] \cdot [\mathsf{T}{:}\mathsf{Z}] \;=\; \deg \mathsf{U} \cdot \deg \mathsf{C} \;=\; \deg \mathsf{E}.$$

This shows that M is a maximal graded subfield of E, cf. §2.1, hence also a maximal graded subfield of D by (2.5).

Furthermore, if $M = \mathsf{L}_0$ is Galois (resp. normal) over F_0, L is Galois (resp. normal) over F. As T is Galois over F by Lemma 2.4, we get that M is Galois (resp. normal) over F. □

For a maximal graded subfield M of D, the field M_0 need not be a maximal subfield of D_0. We therefore modify M to enlarge the degree-0 part, starting with the following observation:

Lemma 3.3. *Let M be a maximal graded subfield of D. Then M_0 is a maximal subfield of D_0 if and only if $\mathsf{M} \supseteq \mathsf{Z}$ and $|\Gamma_{\mathsf{M}} : \Gamma_{\mathsf{Z}}| = \deg \mathsf{C}$. This is the case if $\mathsf{M} \cap \mathsf{C}$ is a maximal graded subfield of C.*

Proof. If M_0 is a maximal subfield of D_0 then $\mathsf{M} \supseteq \mathsf{Z}$ by definition of Z. Hence, we assume $\mathsf{M} \supseteq \mathsf{Z}$ throughout the proof. Then M is a maximal graded subfield of E. We have

$$[\mathsf{M}_0 : \mathsf{Z}_0] \cdot |\Gamma_{\mathsf{M}} : \Gamma_{\mathsf{Z}}| = [\mathsf{M} : \mathsf{Z}] = \deg \mathsf{E} = \deg \mathsf{U} \cdot \deg \mathsf{C} = \deg \mathsf{D}_0 \cdot \deg \mathsf{C}.$$

Hence, M_0 is a maximal subfield of D_0 (i.e. $[\mathsf{M}_0 : \mathsf{Z}_0] = \deg \mathsf{D}_0$) if and only if $|\Gamma_{\mathsf{M}} : \Gamma_{\mathsf{Z}}| = \deg \mathsf{C}$.

Suppose $\mathsf{M} \cap \mathsf{C}$ is a maximal graded subfield of C. Then, since $\mathsf{M} \subseteq C_{\mathsf{E}}(\mathsf{M} \cap \mathsf{C})$, by Lemma 2.4 we have $\Gamma_{\mathsf{M}} \subseteq \Gamma_{C_{\mathsf{E}}(\mathsf{M}\cap\mathsf{C})} = \Gamma_{\mathsf{M}\cap\mathsf{C}} \subseteq \Gamma_{\mathsf{M}}$, hence $|\Gamma_{\mathsf{M}} : \Gamma_{\mathsf{Z}}| = \deg \mathsf{C}$. □

The following Proposition gives the "only if" implication of Theorem 3.1.

Proposition 3.4. *Let M be a maximal graded subfield of D and let T be a maximal graded subfield of C. Then $\mathsf{M}' := (\mathsf{M} \cap C_{\mathsf{D}}(\mathsf{T})) \cdot \mathsf{T}$ is a maximal graded subfield of D for which M'_0 is a maximal subfield of D_0.*

Furthermore, if M is Galois (resp. normal) over F then M' is Galois (resp. normal) over F and M'_0 is Galois (resp. normal) over F_0.

The proof relies on the following lemma:

Lemma 3.5. *Let* A, B *be graded subalgebras of* D *containing* F *such that* A *is a graded field and* B *is contained in* C*. Define* $\mathsf{A}' := \mathsf{A} \cap C_{\mathsf{D}}(\mathsf{B})$*. Then* A/A' *is totally ramified and* $[\mathsf{A} : \mathsf{A}'] \leq [\mathsf{B} : \mathsf{A} \cap \mathsf{B}]$*.*

Proof. Since $\mathsf{B} \subseteq \mathsf{C}$, one has $U = C_{\mathsf{D}}(\mathsf{C}) \subseteq C_{\mathsf{D}}(\mathsf{B})$ and $C_{\mathsf{D}}(\mathsf{B})_0 = \mathsf{U}_0 = \mathsf{D}_0$. Thus, D is totally ramified over $C_{\mathsf{D}}(\mathsf{B})$. Hence i) A/A' is totally ramified (for $\mathsf{A}'_0 = \mathsf{A}_0 \cap C_{\mathsf{D}}(\mathsf{B})_0 = \mathsf{A}_0 \cap \mathsf{D}_0 = \mathsf{A}_0$), and ii) $\Gamma_{\mathsf{A}'} = \Gamma_{\mathsf{A}} \cap \Gamma_{C_{\mathsf{D}}(\mathsf{B})}$ by Lemma 2.1(ii). Since A is a graded field, we have $\mathsf{A} \subseteq C_{\mathsf{D}}(\mathsf{A}) \subseteq C_{\mathsf{D}}(\mathsf{A} \cap \mathsf{B})$. These together yield,

$$\begin{aligned}[\mathsf{A} : \mathsf{A}'] &= |\Gamma_{\mathsf{A}} : \Gamma_{\mathsf{A}'}| = |\Gamma_{\mathsf{A}} : \Gamma_{\mathsf{A}} \cap \Gamma_{C_{\mathsf{D}}(\mathsf{B})}| = |\Gamma_{\mathsf{A}} + \Gamma_{C_{\mathsf{D}}(\mathsf{B})} : \Gamma_{C_{\mathsf{D}}(\mathsf{B})}| \\ &\leq |\Gamma_{C_{\mathsf{D}}(\mathsf{A}\cap\mathsf{B})} : \Gamma_{C_{\mathsf{D}}(\mathsf{B})}| \leq [C_{\mathsf{D}}(\mathsf{A} \cap \mathsf{B}) : C_{\mathsf{D}}(\mathsf{B})] = [\mathsf{B} : \mathsf{A} \cap \mathsf{B}]. \quad \square\end{aligned}$$

Proof of Proposition 3.4. By Lemma 2.4, T is Galois over F. Thus, by Lemma 2.2, T is linearly disjoint from $\mathsf{M} \cap C_{\mathsf{D}}(\mathsf{T})$ over their intersection $(\mathsf{M} \cap C_{\mathsf{D}}(\mathsf{T})) \cap \mathsf{T} = \mathsf{M} \cap \mathsf{T}$. Hence, by definition of M', we have $[\mathsf{M}' : \mathsf{M} \cap C_{\mathsf{D}}(\mathsf{T})] = [\mathsf{T} : \mathsf{M} \cap \mathsf{T}]$. Lemma 3.5, applied with $\mathsf{A} = \mathsf{M}$ and $\mathsf{B} = \mathsf{T}$, states that this dimension is $\geq [\mathsf{M} : \mathsf{M} \cap C_{\mathsf{D}}(\mathsf{T})]$, and therefore implies the first statement. If M is maximal then clearly M' is maximal, and, since M' contains T, M'_0 is a maximal subfield of D_0 by Lemma 3.3.

Assume M is Galois (resp. normal) over F. The application of Lemma 3.5 above also showed that M is totally ramified over $\mathsf{M} \cap C_{\mathsf{D}}(\mathsf{T})$. Hence, by Lemma 2.3, $\mathsf{M} \cap C_{\mathsf{D}}(\mathsf{T})$ is Galois (resp. normal) over F. Since, by Lemma 2.4, T is Galois over F , we get that M' is Galois (resp. normal) over F. Hence, also M'_0 is Galois (resp. normal) over F_0. $\square$

Remark 3.6. For a graded subfield M of D which is not necessarily maximal, the proof gives $[\mathsf{M}' : \mathsf{F}] \geq [\mathsf{M} : \mathsf{F}]$, where $\mathsf{M}' := (\mathsf{M} \cap C_{\mathsf{D}}(\mathsf{T})) \cdot \mathsf{T}$.

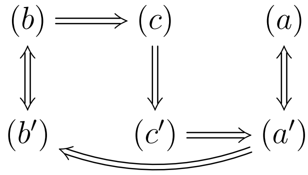
Theorem 3.1 is now proved, and we proceed to deduce Theorem 1.1 from it.

Corollary 3.7. *Let F be a Henselian field, and let D be a tame division algebra with center F. The following are equivalent:*

(a) D has a maximal subfield Galois over F.
(b) $\overline{D}$ has a maximal subfield Galois over $\overline{F}$.
(c) D has a maximal subfield Galois and tamely ramified over F.

Moreover, the list can be extended by the three conditions $(a'), (b'), (c')$ which are obtained from $(a), (b), (c)$ by replacing 'Galois' with 'normal'.

Proof of Corollary 3.7. Trivially, $(a), (b), (c)$ imply $(a'), (b'), (c')$ respectively, and (c') implies (a'). By [13, Lemma 3], (a') implies (a). Since D is tame, $Z(\overline{D})/\overline{F}$ is separable. Therefore, as was noted in [4, Prop. 14.2, p. 59], also (b') implies (b). We will show $(b) \Rightarrow (c)$ and $(a') \Rightarrow (b')$, then the proof is completed:



$(b) \Rightarrow (c)$: Suppose $\overline{D}$ has a maximal subfield Galois over $\overline{F}$. As D is defectless, by Proposition 3.2, $\mathsf{gr}(D)$ has a maximal graded subfield M that is Galois over F. Let M/F be the tame lift of M/F over F (cf. §2.5), i.e. the unique tame Galois extension of F with $\mathsf{gr}(M) = \mathsf{M}$. By the functoriality mentioned in §2.5, this M is a maximal subfield of D.

$(a') \Rightarrow (b')$: Let M be a maximal subfield of D that is normal over F. Then $\mathsf{gr}(M)/\mathsf{gr}(F)$ is normal (cf. §2.5). As D is defectless,

$$[\mathsf{gr}(M)\colon \mathsf{gr}(F)] \;=\; [M\!:\!F] \;=\; \deg D \;=\; \deg \mathsf{gr}(D),$$

showing that $\mathsf{gr}(M)$ is a maximal graded subfield of $\mathsf{gr}(D)$. By Proposition 3.4, $\mathsf{gr}(D)$ has a maximal graded subfield M' normal over F and such that M'_0 is a maximal subfield of $\overline{D} = \mathsf{gr}(D)_0$. By (2.4), M'_0/F is normal. □

4. Tamely ramified noncrossed products

4.1. **Simple residue fields.** It is a fundamental question to determine which division algebras over a given field F are crossed products. As a corollary to Theorem 1.1 we obtain an answer when F is Henselian and division algebras over the residue field $K := \overline{F}$ are sufficiently well understood, e.g. when it is of cohomological dimension 1, cf. [15, Chp. 3].

Corollary 4.1. *Let F be a Henselian field whose residue field K is a local field*[2]*, real closed field, or satisfies $\operatorname{cd} G_K \leq 1$, then every tame central division algebra over F is a crossed product.*

Proof. By Theorem 1.1, it suffices to show that $\overline{D}$ has a maximal subfield which is Galois over $K := \overline{F}$. If $\operatorname{cd} G_K \leq 1$, then $\overline{D} = Z(\overline{D})$ is a field which, since D is tame, is Galois over K. If K is a real closed field, then either K or $K(\sqrt{-1})$ is a maximal subfield of $\overline{D}$ which is Galois over K. If K is a local field then $Z(\overline{D})$ has extensions of arbitrary degree which are Galois over K, simply by composing $Z(\overline{D})$ with unramified extensions of K. This gives the desired result since over local fields every field of degree $\deg \overline{D}$ over $Z(\overline{D})$ is a maximal subfield of $\overline{D}$. □

Note that (i) if K is real closed the assertion can be proved directly without using Corollary 4.1; (2) Examples of fields K for which $\operatorname{cd} G_K \leq 1$ include finite fields, and by Tsen's theorem [12, §19.4], function fields of curves over algebraically closed fields.

4.2. **Global residue fields.** Let Γ be the value group of the valuation on F. We consider next the simplest residue field $K := \overline{F}$ for which noncrossed products exist over F, namely when K is a global field, [3].

[2]We call a field local if it is a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$ for some prime p.

The group $\mathrm{TBr}(F)$ is described by a generalized Witt theorem [1, Proposition 3.5][3] as a direct sum:

$$\mathrm{TBr}(F) \cong \mathrm{Br}(K) \oplus \mathrm{Hom}(G_K, \Delta/\Gamma) \oplus T, \tag{4.1}$$

where G_K is the absolute Galois group, Δ is the divisible hull of Γ, and T is a subgroup consisting of classes of totally ramified division algebras. Moreover, the subgroup of $\mathrm{TBr}(F)$ corresponding to $\mathrm{Br}(K) \oplus \mathrm{Hom}(G_K, \Delta/\Gamma)$ (resp. $\mathrm{Br}(K)$) is the subgroup of classes of inertially split division algebras (resp. inertial division algebras), as described by Witt's theorem [14].

For fixed $\chi \in \mathrm{Hom}(G_K, \Delta/\Gamma)$ and $\eta \in T$, we call the preimage of $\chi + \eta$ under (4.1) the fiber over $\chi + \eta$. To describe the location of noncrossed products in $\mathrm{TBr}(F)$, we ask which χ and η the fiber over $\chi + \eta$ contains noncrossed products? This problem was answered in [6] and [5] for the inertially split subgroup, i.e. when $\eta = 0$. In the following we combine Theorem 1.1 with the methods of [6] and [5] to answer this problem for the entire group $\mathrm{TBr}(F)$.

To this end, we fix χ and η and let $\mathfrak{C}$ be the fiber over $\chi + \eta$. By Lemma 2.5, $Z = Z_{\mathfrak{C}} := Z(\overline{c})$ and $e = e_{\mathfrak{C}} := \deg c / \deg \overline{c}$ are independent of $c \in \mathfrak{C}$.

Note that Z/K is abelian, as division algebras in $\mathfrak{C}$ are tame. If Z/K is cyclic, we say that it is of *infinite height* if for every integer m, Z/K embeds into a cyclic extension L/K with $[L : Z] = m$.

Theorem 4.2. *Let Z'/K be the maximal subextension of Z/K of order prime to $l := \mathrm{char}\, K$. Then $\mathfrak{C}$ consists of crossed products if and only if Z'/K is cyclic of infinite height.*

Moreover, we will show that if $\mathfrak{C}$ contains one noncrossed product it contains infinitely many of them.

Theorem 4.2 was known if $\mathfrak{C}$ consists of inertially split division algebras by [6] and [5]. Furthermore, [6], [5], prove the existence of index bounds which separate crossed and noncrossed products in each fiber. We do not know if such bounds exist in fibers which are not inertially split. Nevertheless, we prove that there is a number m such that there are noncrossed product of every index dividing m in $\mathfrak{C}$ (Remark 4.7).

4.3. **Residue classes, Galois covers, and their local degrees.** For $m \in \mathbb{N}$, let $\mathfrak{C}_m$ be the set of $c \in \mathfrak{C}$ with $\deg c = me$, and $\overline{\mathfrak{C}}_m$ (resp. $\overline{\mathfrak{C}}$) the set of residue classes of $\mathfrak{C}_m$ (resp. $\mathfrak{C}$). Note that by Lemma 2.5, $\overline{\alpha^F + c} = \alpha^Z + \overline{c} \in \mathrm{Br}(Z)$ for all $\alpha \in \mathrm{Br}(K), c \in \mathrm{Br}(F)$. Hence, for any $\beta \in \overline{\mathfrak{C}}_m$,

$$\overline{\mathfrak{C}}_m = \{\alpha^Z + \beta : \alpha \in \mathrm{Br}(K), \deg(\alpha^Z + \beta) = m\}. \tag{4.2}$$

By Theorem 1.1 the following conditions are equivalent:

(A_m) $\mathfrak{C}_m$ consists of crossed products;

[3] The decomposition of $\mathrm{TBr}(F)$ is described in [1] on the level of primary components.

($\overline{A}_m$) Every class in $\overline{\mathfrak{C}}_m$ has a splitting field L, Galois over K, with $[L : Z] = m$.

We call $L \supseteq Z$ an *m-cover* of Z/K if L is Galois over K and $[L : Z] = m$. The cover L is *cyclic* if L/K is cyclic. For a prime $\mathfrak{p}$ of K, let $K_\mathfrak{p}$ denote the completion at $\mathfrak{p}$ and let $[L : Z]_\mathfrak{p} := [L_{\mathfrak{P}'} : Z_{\mathfrak{P}' \cap Z}]$ for any prime $\mathfrak{P}'$ of L dividing $\mathfrak{p}$.

Let $\beta \in \mathrm{Br}(Z)$. Recall that for a prime $\mathfrak{P}$ of Z, the degree $\deg_\mathfrak{P} \beta$ of $\beta^{Z_\mathfrak{P}}$ is equal to its exponent $\exp_\mathfrak{P} \beta$. By the Albert-Brauer-Hasse-Noether theorem

$$\text{(4.3)} \qquad \beta \text{ is split by } L \text{ if and only if } \deg_\mathfrak{P} \beta \mid [L : Z]_\mathfrak{P} \text{ for every prime } \mathfrak{P} \text{ of } Z.$$

This allows to translate ($\overline{A}_m$) into conditions on the local degrees of covers, as follows. Let $d_\mathfrak{p}(m) := m$ for every finite prime $\mathfrak{p}$ of K, $d_\mathfrak{p}(m) = \gcd(m, 2)$ for every real prime $\mathfrak{p}$ of K which is unramified in Z, and $d_\mathfrak{p}(m) := 1$ otherwise.

Let S be a finite set of primes of K. An m-cover L of Z has *full local degree in* S if $[L : Z]_\mathfrak{p} = d_\mathfrak{p}(m)$ for all $\mathfrak{p} \in S$.

Proposition 4.3. *Assume $\mathfrak{C}_m \neq \emptyset$. There exists a finite set T of primes of K such that (B_m) $\Rightarrow$ ($\overline{A}_m$) $\Rightarrow$ (B'_m), where (B_m) and (B'_m) are the following conditions:*

(B_m) *for every S, Z/K has an m-cover with full local degree in S.*

(B'_m) *for every S disjoint from T, Z/K has an m-cover with full local degree in S.*

Proof. (B_m) $\Rightarrow$ ($\overline{A}_m$): Let $\beta \in \overline{\mathfrak{C}}_m$, and S the set of primes $\mathfrak{p}$ of K such that the restriction of β to $Z_\mathfrak{P}$ is nontrivial for some prime $\mathfrak{P} \mid \mathfrak{p}$ of Z. Applying (B_m) to S, we obtain an m-cover L of Z with full local degree in S. By (4.3), L splits β, as required.

($\overline{A}_m$) $\Rightarrow$ (B'_m): For every $p \mid [Z : K]$, let $\mathfrak{p}_1^{(p)}, \mathfrak{p}_2^{(p)}, \ldots$ be any enumeration of the primes of K so that $p^n \mid [Z : K]_{\mathfrak{p}_{i+1}^{(p)}}$ implies $p^n \mid [Z : K]_{\mathfrak{p}_i^{(p)}}$ for all $i, n \in \mathbb{N}$. Let T be the set $\{\mathfrak{p}_1^{(p)}, \mathfrak{p}_2^{(p)} \mid p \text{ divides } m\}$.

Let S be disjoint from T and $\beta \in \overline{\mathfrak{C}}_m$. Define S' to be the subset of primes $\mathfrak{p} \in S$ for which $\deg_\mathfrak{P} \beta < d_\mathfrak{p}(m)$ for all $\mathfrak{P} \mid \mathfrak{p}$. Since S is disjoint from T, we can apply [5, Lemma 2.5] to obtain a class $\alpha \in \mathrm{Br}(K)$ such that $\deg \alpha^Z = m$, and $\deg_\mathfrak{P} \alpha^Z = d_\mathfrak{p}(m)$ for all $\mathfrak{p} \in S', \mathfrak{P} \mid \mathfrak{p}$. Furthermore, the proof of [5, Lemma 2.5] gives $\deg_\mathfrak{P} \alpha^Z = 1$ for all $\mathfrak{P} \mid \mathfrak{p}, \mathfrak{p} \in S \setminus S'$.

Let $\gamma := \alpha^Z + \beta$. Since $\exp_\mathfrak{P} \beta < \exp_\mathfrak{P} \alpha^Z = d_\mathfrak{p}(m)$ for all $\mathfrak{P} \mid \mathfrak{p}, \mathfrak{p} \in S'$, we have $\deg_\mathfrak{P} \gamma = \exp_\mathfrak{P} \gamma = d_\mathfrak{p}(m)$ for all $\mathfrak{P} \mid \mathfrak{p}, \mathfrak{p} \in S'$. For every $\mathfrak{p} \in S \setminus S'$ there is a prime $\mathfrak{P} \mid \mathfrak{p}$, such that $\deg_\mathfrak{P} \beta = d_\mathfrak{p}(m)$, and hence, as $\deg_\mathfrak{P} \alpha^Z = 1$, one has $\deg_\mathfrak{P} \gamma = d_\mathfrak{p}(m)$. Enlarging S, we may assume S' contains a finite prime $\mathfrak{p}$ and hence that $\deg \gamma = \deg_\mathfrak{P} \gamma = m$, where $\mathfrak{P} \mid \mathfrak{p}$.

By applying (A_m) to γ, we obtain an m-cover of Z/K which splits γ. Thus, by (4.3), L has full local degree in S, as required. □

Remark 4.4. Assume $\mathfrak{C}_m \neq \emptyset$. The proof reveals that if (B'_m) fails then there are in fact infinitely many noncrossed products in $\mathfrak{C}_m$. Indeed if (B'_m) fails for

S, it fails for every set S' which contains S and is disjoint from T. Since the constructed $\gamma \in \overline{\mathfrak{C}}_m$ has nontrivial completions at primes of S', there are infinitely many $\gamma \in \overline{\mathfrak{C}}_m$ for which $(\overline{A}_m)$ fails. Thus, there are infinitely many noncrossed products in $\mathfrak{C}_m$.

4.4. **Proof of Theorem 4.2.** Set $l = \mathrm{char}(K)$. We shall use the following lemmas:

Lemma 4.5. *(*[5, Lemma 2.12]*) There is a finite set S_0 such that every p^n-cover L with full local degree in S_0 has abelian kernel $A = \mathrm{Gal}(L/Z)$, for which the conjugation action of $\mathrm{Gal}(Z/Z')$ on A is trivial.*

Lemma 4.6. *Let $p \neq l$ be a prime. There exists a finite set S_0 disjoint from T such that every p^n-cover L of Z/K with full local degree in S_0 contains a p^n-cover L' of Z'/K.*

Furthermore, if L has full local degree in a set $S \supseteq S_0$ then so does L'.

Proof. Let S_0 be as in Lemma 4.5 and let $B_l = \mathrm{Gal}(Z/Z')$. Let $A_l = \mathrm{Gal}(L/Z')$. Since $|A|$ and $|B_l|$ are relatively prime, the group extension

$$1 \to A \to A_l \to B_l \to 1$$

is split by the Schur-Zassenhaus theorem. Since B_l acts trivially on A, $A_l = A \oplus \hat{B}_l$ with $\hat{B}_l \cong B_l$. In particular, A_l is abelian. Letting $G = \mathrm{Gal}(L/K)$ and $B = \mathrm{Gal}(Z'/K)$ the group extension $1 \to A_l \to G \to B \to 1$ induces an action of B on A_l. Being a characteristic subgroup of A_l, $\hat{B}_l$ is B-invariant, hence normal in G. The fixed field $L' \subseteq L$ of $\hat{B}_l$ is then a cover of Z'/K with associated group extension

$$1 \to A_l/\hat{B}_l \to G/\hat{B}_l \to B \to 1.$$

Since $A_l/\hat{B}_l \cong A$ it is a p^n-cover.

Full local degree in S is inherited from L since the completions of L' and of Z are linearly disjoint over a completion of Z'. □

Proof of Theorem 4.2. Fix $\beta \in \overline{\mathfrak{C}}$ and let $m = \deg \beta$. By (4.2) we see that $\mathfrak{C}_{m'} \neq \emptyset$ for every $m \,|\, m'$. For a prime $p \neq l$, let p^{s_p} (resp. 2^{r_2} if $p = 2$) denote the number of p-power (resp. 2-power) roots of unity in Z (resp. in $Z(\sqrt{-1})$).

Assume Z'/K is cyclic of finite height or noncyclic. We first claim that there is a prime p for which (B'_{p^n}) fails for all sufficiently large n. Assume first that Z'/K is noncyclic and let $p \neq l$ be a prime for which the p-Sylow subgroup of $\mathrm{Gal}(Z/K)$ is noncyclic. By [5, Proposition 3.3], (B'_{p^n}) fails for all $n > 2s_p$ if p is odd and for $n > 2(r_2 + 2)$ if $p = 2$.

Assume next that Z'/K is cyclic of finite height, and fix $m \in \mathbb{N}$ such that Z'/K has no cyclic m-cover. We can further assume that m is a prime power. Indeed, writing $m = \prod_p p^{n_p}$, p prime, if there are cyclic p^{n_p}-covers for all $p \,|\, m$, their composite gives a cyclic m-cover of Z'/K. Let $m = p^{n_p}$ and let $n > n_p + s_p + 1$. By [6, Theorem 6.4][4], there is a set S disjoint from T, such that Z'/K has no p^n-cover

[4]If Z'/K is non-exceptional we can choose $n := n_p + s_p + 1$, see [6].

with full local degree in S. By Lemma 4.6, there is a finite set S_0 such that Z/K has no p^n-cover L with full local degree in $S_0 \cup S$. Hence, (B'_{p^n}) fails, proving the claim.

Fix an n for which (B'_{p^n}) fails and such that p^n is at least the largest p-power dividing m. Letting $m' = \operatorname{lcm}(m, p^n)$, by Lemma 4.5 every m'-cover with full local degree in a set $S \supseteq S_0$ has an abelian kernel and hence contains a p^n-cover with full local degree in S. Hence, $(B'_{m'})$ fails. As $\mathfrak{C}_{m'} \neq \emptyset$, Proposition 4.3 implies that $(A_{m'})$ fails.

Conversely, assume that Z'/K is cyclic of infinite height. By [6, Theoerem 6.3], for every m prime to l, Z'/K has an m-cover with full local degree. Taking composites with Z, we get that (B_m) holds for every m prime to l. By [5, Lemma 2.10], (B_{l^n}) holds for all n. Taking composites, we see that (B_m) holds for all $m \in \mathbb{N}$. Thus, by Proposition 4.3, (A_m) holds for all m. □

Remark 4.7. Assuming $\mathfrak{C}$ does not consist of crossed products, the proof reveals that there is $p^{n_p} \in \mathbb{N}$, p prime, such that (A_m) fails for every $m \in \mathbb{N}$ with $\mathfrak{C}_m \neq \emptyset$ and $p^{n_p} \mid m$. By Remark 4.4, for such m, $\mathfrak{C}_m$ contains infinitely many noncrossed products. If the p-Sylow subgroup of $\operatorname{Gal}(Z/K)$ is noncyclic and $p \neq l$, we can choose $n_p = 2s_p + 1$ if p is odd and $n_2 = 2(r_2 + 2) + 1$ if $p = 2$. If Z'/K is cyclic with no cyclic p^{k_p}-cover, we can choose $n_p = k_p + s_p + 2$.

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