

ON SEMIABELIAN p -GROUPS

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ABSTRACT. The family of semiabelian p -groups is the minimal family that contains $\{1\}$ and is closed under quotients and semidirect products with finite abelian p -groups. Kisilevsky and Sonn have solved the minimal ramification problem for a family \mathcal{G}_p of p -groups that is contained in the family of semiabelian p -groups. We show that \mathcal{G}_p is in fact the entire family of semiabelian p -groups and by this complete their solution to all semiabelian p -groups.

1. INTRODUCTION

This paper is motivated by the minimal ramification problem for p -groups. Given a p -group G it is an open problem to find the minimal number of primes ramified in a G -extension of \mathbb{Q} (see [5]). As a consequence of Minkowski's Theorem this number is greater or equal to $d(G)$, the minimal number of generators of G . In [3], Kisilevsky and Sonn proved this number is exactly $d(G)$ for a family of p -groups denoted by \mathcal{G}_p and defined as follows:

Definition 1.1. Let \mathcal{G}_p be the minimal family that satisfies:

- (1) any abelian p -group is in \mathcal{G}_p ,
- (2) if $H, G \in \mathcal{G}_p$ then the standard wreath product $H \wr G$ is also in \mathcal{G}_p ,
- (3) if $G \in \mathcal{G}_p$ and $G \rightarrow \Gamma$ is a rank preserving epimorphism, i.e. with $d(G) = d(\Gamma)$, then $\Gamma \in \mathcal{G}_p$.

This family is contained in the family of semiabelian groups (see [3]):

Definition 1.2. The family of *semiabelian* groups \mathcal{S} is the minimal family of groups that satisfies:

- (1) $\{1\} \in \mathcal{S}$,
- (2) if A is a finite abelian group and $H \in \mathcal{S}$ acts on A then the semidirect product $A \rtimes H$ is in \mathcal{S} ,
- (3) if $G \in \mathcal{S}$ and $H \triangleleft G$ then $G/H \in \mathcal{S}$.

We shall prove that \mathcal{G}_p is precisely the family of semiabelian p -groups. In fact this is an immediate corollary of the following theorem:

Theorem 1.3. *Let G be a semiabelian p -group. Then there are abelian p -groups A_1, A_2, \dots, A_r for which there is a rank preserving epimorphism $A_1 \wr (A_2 \wr \dots \wr A_r) \rightarrow G$.*

By this we complete the solution of the minimal ramification problem for semiabelian p -groups.

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2. PROPERTIES OF DECOMPOSITIONS

The family of semiabelian groups has appeared in many forms in problems that arise from field theory (e.g. geometric realizations [1],[4],[6], generic extensions [6] and the minimal ramification problem [3]). The following notion of a decomposition is used in [1] to characterize semiabelian groups:

Definition 2.1. Let G be a non-trivial group. A decomposition of G is an abelian normal subgroup $A \triangleleft G$ and a proper subgroup $H < G$ such that $G = AH$.

Dentzer ([1]) showed that a non-trivial group G is semiabelian if and only if there is a decomposition $G = AH$ where H is semiabelian.

In the following Lemma we summarize several properties of such decompositions which we shall use repeatedly. Let $\Phi(G)$ denote the Frattini subgroup of a group G . Recall that if G is a p -group then $\Phi(G) = G^p[G, G]$.

Lemma 2.2. *Let G be a p -group with decomposition $G = AH$. Let $\bar{A} = A/A^p[A, H]$, $\pi : A \rightarrow \bar{A}$ the natural map and M a minimal subgroup of A for which $\pi(M) = \bar{A}$. Then:*

- (1) $[A, H]$ is a subgroup of A that is normal in G ,
- (2) $\Phi(G) = A^p[A, H]\Phi(H)$,
- (3) \bar{A} is a non-trivial elementary abelian p -group and $d(\bar{A}) = d(M)$.

If we assume in addition that $A \cap H \subseteq A^p[A, H] \cap \Phi(H)$, then:

- (4) there is an isomorphism:

$$\psi : G/\Phi(G) \cong \bar{A} \times (H/\Phi(H)),$$

that is given explicitly for all $a \in A, h \in H$ by:

$$\psi(ah\Phi(G)) = (aA^p[A, H], h\Phi(H)).$$

- (5) $d(G) = d(\bar{A}) + d(H)$.

Proof. (1) For $a \in A, h \in H$, we have $[a, h] = a^{-1}h^{-1}ah = a^{-1}a^h$ where $a^h := h^{-1}ah \in A$ and hence $[A, H] \leq A$.

Since $[A, H]$ is a subgroup of A it is centralized by A . For $h, h' \in H$ and $a \in A$ we have $[a, h]^{h'} = [a^{h'}, h^{h'}] \in [A, H]$ and hence $[A, H]^{h'} \subseteq [A, H]$. It follows that $[A, H]$ is normalized by A and H and hence is a normal subgroup of G .

- (2) Clearly

$$A^p[A, H]\Phi(H) = A^p[A, H]H^p[H, H] \subseteq G^p[G, G] = \Phi(G).$$

To show the converse we prove that for $g_1 = a_1h_1, g_2 = a_2h_2 \in G$, where $a_1, a_2 \in A$ and $h_1, h_2 \in H$, the commutator $[g_1, g_2] = g_1^{-1}g_2^{-1}g_1g_2$ and g_1^p are elements of $A^p[A, H]\Phi(H)$. We use the following identities:

$$[x, yz] = [x, z][x, y]^z,$$

$$[xy, z] = [x, z]^y[y, z],$$

where $x^y = y^{-1}xy$. It follows that:

$$[g_1, g_2] = [a_1h_1, g_2] = [a_1, g_2]^{h_1}[h_1, g_2].$$

Expanding the commutators on the right hand side we have:

$$[a_1, g_2] = [a_1, a_2 h_2] = [a_1, h_2][a_1, a_2]^{h_2} = [a_1, h_2] \in [A, H],$$

$$[h_1, g_2] = [h_1, a_2 h_2] = [h_1, h_2][h_1, a_2]^{h_2} \in [H, H][A, H]^{h_2}.$$

Since by part (1), $[A, H] \triangleleft G$ we have:

$$[g_1, g_2] = [a_1, h_2]^{h_1}[h_1, h_2][h_1, a_2]^{h_2} \in [A, H][H, H][A, H] = [A, H][H, H].$$

We therefore have $[G, G] \subseteq [A, H][H, H]$. To prove that $g_1^p \in A^p[A, H]\Phi(H)$ we use the equality:

$$g_1^p = a_1^p h_1^p \pmod{[G, G]}.$$

It follows that $g_1^p \in A^p H^p [G, G] \subseteq A^p [A, H] \Phi(H)$.

- (3) Since all p -powers of A are in $A^p [A, H]$, \bar{A} is an elementary abelian p -group. Assume on the contrary $\bar{A} = \{1\}$. Then A is contained in $\Phi(G)$ and hence the equality $G = AH$ implies $G = H$. This contradicts the assumption that H is a proper subgroup of G as part of a decomposition.

Let us show that $d(\bar{A}) = d(M)$. Let $\bar{A} = \langle \bar{a}_1, \dots, \bar{a}_d \rangle$ where $d := d(\bar{A})$. Since $\pi(M) = \bar{A}$, $d(M) \geq d(\bar{A})$. Furthermore, M contains elements a_1, \dots, a_d such that $\pi(a_i) = \bar{a}_i$ for $i = 1, \dots, d$. The subgroup $M' := \langle a_1, \dots, a_d \rangle \leq M$ maps under π onto \bar{A} and hence by the minimality of M , $M = M'$. It follows that $d(M) = d$.

- (4) Let us first prove $G/\Phi(G) \cong \bar{A} \times H/\Phi(H)$ under the assumption $A \cap H = \{1\}$. In such case $G = A \times H$. Since by part (1) $A^p [A, H] \leq A$, we have $A^p [A, H] \cap \Phi(H) = \{1\}$ and hence part (2) shows

$$\Phi(G) = A^p [A, H] \rtimes \Phi(H).$$

We therefore have:

$$(2.1) \quad G/\Phi(G) = (A \rtimes H) / (A^p [A, H] \rtimes \Phi(H)).$$

As A^p is characteristic in A we have $A^p \triangleleft G$. Together with part (1) this implies that $A^p [A, H]$ is a normal subgroup of G . Therefore the right hand side of (2.1) is isomorphic to:

$$(2.2) \quad \begin{aligned} & (A \rtimes H / A^p [A, H] \rtimes \{1\}) / (A^p [A, H] \rtimes \Phi(H) / A^p [A, H] \rtimes \{1\}) \\ & \cong (A / A^p [A, H] \rtimes H) / (\{1\} \rtimes \Phi(H)) = (\bar{A} \times H) / (1 \times \Phi(H)). \end{aligned}$$

Since $[A, H] = \{a^{-1}a^h \mid a \in A, h \in H\}$, H acts trivially on $A/[A, H]$ and hence the actions in the semidirect products in (2.2) are trivial. We therefore have an isomorphism:

$$G/\Phi(G) \cong (\bar{A} \times H) / (\{1\} \times \Phi(H)) \cong \bar{A} \times (H/\Phi(H)).$$

Let us prove the assertion without assuming further that $A \cap H = \{1\}$. Note that $A \cap H$ is a normal subgroup of G and let $G_0 = G/A \cap H$, $A_0 = A/A \cap H$, $H_0 = H/A \cap H$. Since $A_0 \cap H_0 = \{1\}$ we have $G_0 = A_0 \rtimes H_0$. In particular, the assertion holds for the decomposition $G_0 = A_0 H_0$. Thus,

$$(2.3) \quad G_0/\Phi(G_0) \cong \bar{A}_0 \times H_0/\Phi(H_0),$$

where $\bar{A}_0 = A_0/A_0^p[A_0, H_0]$. Since,

$$(2.4) \quad \begin{aligned} \Phi(H_0) &= H_0^p[H_0, H_0] = H^p[H, H]/A \cap H = \Phi(H)/A \cap H, \\ A_0^p[A_0, H_0] &= A^p[A, H]/A \cap H, \end{aligned}$$

we have:

$$\begin{aligned} \bar{A}_0 &= (A/A \cap H)/(A^p[A, H]/A \cap H) \cong A/A^p[A, H] = \bar{A}, \\ H_0/\Phi(H_0) &= (H/A \cap H)/(\Phi(H)/A \cap H) \cong H/\Phi(H). \end{aligned}$$

By part (2) we also have:

$$\Phi(G_0) = A_0^p[A_0, H_0]\Phi(H_0) = A^p[A, H]\Phi(H)/A \cap H = \Phi(G)/A \cap H.$$

We therefore have:

$$\begin{aligned} G/\Phi(G) &\cong (G/A \cap H)/(\Phi(G)/A \cap H) = G_0/\Phi(G_0) \\ &\cong \bar{A}_0 \times (H_0/\Phi(H_0)) \cong \bar{A} \times H/\Phi(H). \end{aligned}$$

Since in all of the above isomorphisms the coset of $a \in A$ (resp. $h \in H$) passes to the coset of a (resp. h) in the image, the resulting isomorphism

$$\psi : G/\Phi(G) \rightarrow \bar{A} \times H/\Phi(H)$$

is given for all $a \in A, h \in H$ by:

$$\psi(ah\Phi(G)) = (aA^p[A, H], h\Phi(H)).$$

- (5) By part (4) we have $d(G/\Phi(G)) = d(\bar{A}) + d(H/\Phi(H))$. Recall that the Frattini subgroup satisfies $d(\Gamma) = d(\Gamma/\Phi(\Gamma))$ for any p -group Γ . Therefore $d(G) = d(\bar{A}) + d(H)$. □

By iterating Lemma 2.2.(4) we have:

Corollary 2.3. *Let H_0 be a p -group. For $i = 1, \dots, k$, let $H_{i-1} = A_i H_i$ be a decomposition for which $A_i \cap H_i \subseteq A_i^p[A_i, H_i] \cap \Phi(H_i)$ and let $\bar{A}_i := A_i/A_i^p[A_i, H_i]$. Then there is an isomorphism*

$$\psi : H_0/\Phi(H_0) \cong \left(\prod_{i=1}^k \bar{A}_i \right) \times H_k/\Phi(H_k)$$

such that for all $a_i \in A_i, i = 1, \dots, k$, and $h \in H_k$:

$$(2.5) \quad \psi(a_1 \dots a_k h \Phi(H_0)) = (a_1 A_1^p[A_1, H_1], \dots, a_k A_k^p[A_k, H_k], h \Phi(H_k)).$$

Note that since $H_0 = A_1 \dots A_k H_k$, every element $x \in H_0$ can be written as $a_1 \dots a_k h$ for $a_i \in A_i, i = 1, \dots, k, h \in H_k$ and hence (2.5) provides a description of ψ for all elements of $H_0/\Phi(H_0)$.

3. MINIMAL DECOMPOSITIONS

A key ingredient in our proof of Theorem 1.3 is to find a decomposition $G = AH$ such that $d(G) = d(\bar{A}) + d(H)$, where $\bar{A} = A/A^p[A, H]$. We shall prove this is the case for minimal decompositions.

Definition 3.1. Let G be a semiabelian group. A *minimal decomposition* of G is:

(1) a minimal normal abelian subgroup $A \triangleleft G$ for which there is a semiabelian proper subgroup H' of G satisfying $G = AH'$,

(2) a minimal subgroup $H \leq G$ for which H is semiabelian and $G = AH$ (for the same A given in (1)).

By Dentzer's result ([1]) any non-trivial semiabelian group G has a decomposition and therefore also a minimal one. We first prove that such decompositions satisfy the conditions of Lemma 2.2.(4):

Proposition 3.2. *Let G be a semiabelian p -group with a minimal decomposition $G = AH$. Then $A \cap H \subseteq A^p[A, H] \cap \Phi(H)$.*

Proof. We divide the proof into two parts: (1) $A \cap H \subseteq A^p[A, H]$, (2) $A \cap H \subseteq \Phi(H)$.

(1) Let $\pi_A : A \rightarrow \bar{A}$ where $\bar{A} := A/A^p[A, H]$. Assume on the contrary there is an $a \in A \cap H$ with non-trivial image $\bar{a} := \pi_A(a)$. By Lemma 2.2.(3), \bar{A} is an elementary abelian p -group and hence can be viewed as an \mathbb{F}_p -vector space. We can choose an \mathbb{F}_p -subspace $B_1 \subseteq \bar{A}$ such that $\bar{A} = \langle \bar{a} \rangle \oplus B_1$. The group $A_1 := \pi_A^{-1}(B_1)$ is a proper subgroup of A which is normal in G . We also have $\pi_A(A_1)\pi_A(A \cap H) = \bar{A}$ and therefore $A_1(A \cap H)A^p[A, H] = A$. Since $A_1 \supseteq A^p[A, H]$ this implies $A_1(A \cap H) = A$ and therefore $A_1H = A_1(A \cap H)H = AH = G$. The equality $G = A_1H$ contradicts the minimality of A .

(2) Let us show that $A \cap H \subseteq \Phi(H)$ by induction on $|G|$. Assume that for any semiabelian group G_0 with $|G_0| < |G|$ and any minimal decomposition $G_0 = BK$, we have $B \cap K \subseteq \Phi(K)$.

Let $\pi_H : H \rightarrow H/\Phi(H)$ and assume on the contrary there is an $a \in A \cap H$ for which $\hat{a} := \pi_H(a)$ is non-trivial. Let $H_1 = H$ and $H_i = A_{i+1}H_{i+1}$, $i = 1, 2, \dots, k-1$, be a sequence of minimal decompositions such that H_k is the first for which $\hat{a} \notin \pi_H(H_k)$.

Let $\bar{A}_i = A_i/A_i^p[A_i, H_i]$ and $\pi_i : A_i \rightarrow \bar{A}_i$. By the induction hypothesis and part 1, $A_i \cap H_i \subseteq A_i^p[A_i, H_i] \cap \phi(H_i)$, for $i = 2, \dots, k$. Thus, we can apply Corollary 2.3 and obtain an isomorphism

$$\psi : H/\Phi(H) \cong \left(\prod_{i=2}^k \bar{A}_i \right) \times H_k/\Phi(H_k).$$

such that for all $a_i \in A_i, i = 2, \dots, k, h \in H_k$:

$$(3.1) \quad \psi(\pi_H(a_2 \dots a_k h)) = (\pi_2(a_2), \dots, \pi_k(a_k), \pi_H(h)).$$

As $H_{k-1} = A_k H_k$, (3.1) implies:

$$(3.2) \quad \begin{aligned} \psi(\pi_H(H_{k-1})) &= \{1\}^{k-2} \times \bar{A}_k \times H_k/\Phi(H_k), \\ \psi(\pi_H(H_k)) &= \{1\}^{k-1} \times H_k/\Phi(H_k). \end{aligned}$$

Write $a = a_2 a_3 \dots a_k h$ for $a_i \in A_i$ and $h \in H_k$, $i = 2, \dots, k$. Since $\hat{a} \in \pi_H(H_{k-1}) \setminus \pi_H(H_k)$, (3.2) implies:

$$\psi(\hat{a}) \in (\{1\}^{k-2} \times \bar{A}_k \times H_k/\Phi(H_k)) \setminus (\{1\}^{k-1} \times H_k/\Phi(H_k)),$$

and hence by (3.1), $\pi_k(a_k) \neq 1$.

Let $\pi_k(a_k), x_1, \dots, x_r$ be a basis of \overline{A}_k and let $A'_k := \pi_k^{-1}(\langle x_1, \dots, x_r \rangle)$. Then A'_k is a proper subgroup of A_k which is normal in H_{k-1} . Since $\langle \pi_k(a_k) \rangle \pi_k(A'_k) = \overline{A}_k$ and as $A'_k \supseteq A_k^p[A_k, H_k]$, we have $\langle a_k \rangle A'_k = A_k$.

The group $U_{k-1} := A'_k H_k$ is a semiabelian subgroup of H_{k-1} . Since A'_k is a proper subgroup of A_k and A_k is minimal we deduce that U_{k-1} is a proper subgroup of H_{k-1} . Iteratively, define a semiabelian subgroup $U_i := A_{i+1} U_{i+1}$ of H_i for $i = 1, \dots, k-2$. The decompositions $H_k = A_{i+1} H_{i+1}$ are minimal and hence each U_i is a proper subgroup of H_i for $i = 1, \dots, k-1$.

We now claim that $AU_1 = G$. We have:

$$(3.3) \quad AU_1 = AA_2 \dots A_{k-1} A'_k H_k = AA_2 \dots A_{k-1} H_k A'_k H_k.$$

Since

$$a_k = a_{k-1}^{-1} \dots a_2^{-1} a h^{-1} \in A_{k-1} \dots A_2 A H_k = AA_2 \dots A_{k-1} H_k,$$

the right hand side of (3.3) contains:

$$AA_2 \dots A_{k-1} \langle a_k \rangle A'_k H_k = A_1 \dots A_k H_k = G.$$

It follows that $G = AU_1$ which contradicts the minimality of $H_1 = H$. \square

Combining Lemma 2.2 and Proposition 3.2 we have:

Theorem 3.3. *Let $G = AH$ be a minimal decomposition of a semiabelian p -group G . Then $d(G) = d(\overline{A}) + d(H)$. In particular $d(H) < d(G)$.*

Proof. By Proposition 3.2, $A \cap H \subseteq A^p[A, H] \cap \Phi(H)$. By Lemma 2.2.(5) this implies $d(G) = d(\overline{A}) + d(H)$. By Lemma 2.2.(3), \overline{A} is non-trivial and hence $d(H) < d(G)$. \square

4. EQUALITY OF FAMILIES

We prove Theorem 1.3 in three steps.

Step I. At first, we use Theorem 3.3 to prove the following Corollary:

Corollary 4.1. *Let G be a semiabelian group with a minimal decomposition $G = AH$. Let $G_0 := A \rtimes H$ with respect to the action induced by conjugation in G . Then there is a rank-preserving epimorphism $\psi_1 : G_0 \rightarrow G$.*

Proof. Let $\psi_1 : G_0 \rightarrow G$ be defined by $\psi_1(a, h) = ah$. It is an homomorphism since for all $a_i \in A, h_i \in H, i = 1, 2$,

$$\psi_1(a_1, h_1) \psi_1(a_2, h_2) = a_1 h_1 a_2 h_2 = a_1 a_2^{h_1^{-1}} h_1 h_2 = \psi_1((a_1, h_1)(a_2, h_2)).$$

Since A and H are in $\text{Im}(\psi)$, ψ is surjective. Let $A_0 = A \rtimes 1, H_0 = 1 \rtimes H \leq G_0$ and $\overline{A}_0 = A_0/A_0^p[A_0, H_0]$. Since

$$A_0 \cap H_0 = \{1\} \subseteq A_0^p[A_0, H_0] \cap \Phi(H_0),$$

it follows from Lemma 2.2.(5) that $d(G_0) = d(\overline{A}_0) + d(H_0)$. We have $H_0 \cong H$ and since

$$[A_0, H_0] = \langle a^{-1} a^h \mid a \in A_0, h \in H_0 \rangle \cong [A, H],$$

we also have $\overline{A}_0 \cong \overline{A}$. In particular, $d(G_0) = d(\overline{A}) + d(H)$. By Theorem 3.3 $d(G) = d(\overline{A}) + d(H)$ and hence $d(G_0) = d(G)$. \square

Step II. We show that the group G_0 in Corollary 4.1 is a rank preserving epimorphic image of a corresponding wreath product. We first recall the following definition:

Definition 4.2. Let H be a group and A an abelian group.

- (1) Let $M_1^H(A)$ be the induced H -module, i.e. the abelian group of all functions $f : H \rightarrow A$ with the H -action $f^h(x) = f(xh^{-1})$ for all $x, h \in H, f \in M_1^H(A)$.
- (2) For every $a \in A$, let $a_* \in M_1^H(A)$ be defined by $a_*(1) = a$ and $a_*(h) = 1$ for $h \neq 1$.
- (3) The wreath product $A \wr H$ is the semidirect product $M_1^H(A) \rtimes H$ with respect to the above H -action.

Note that for $a \in A, h \in H$, one has $a_*^h(x) = a$ if $x = h$ and $a_*^h(x) = 1$ otherwise. Since any $f \in M_1^H(A)$ can be written as $\prod_{h \in H} f(h)_*^h$, one has:

$$(4.1) \quad M_1^H(A) = \langle a_*^h | a \in A, h \in H \rangle.$$

To compute the rank of wreath products we shall use the following lemma:

Lemma 4.3. *Let A and H be p -groups and assume A is abelian. Then*

$$d(A \wr H) = d(A) + d(H).$$

Proof. Let $G := A \wr H$ and $\overline{M_1^H(A)} = M_1^H(A)/[M_1^H(A), H]$. By Lemma 2.2.(5):

$$(4.2) \quad d(G) = d(\overline{M_1^H(A)}) + d(H).$$

Define the norm homomorphism $N : M_1^H(A) \rightarrow A$ by $N(f) = \prod_{h \in H} f(h)$. Since $N(a_*) = a$, N is an epimorphism. By (4.1) we have:

$$\begin{aligned} [M_1^H(A), H] &= \langle f^{-1}f^h | f \in M_1^H(A), h \in H \rangle \\ &= \langle a_*^{-1}a_*^h | a \in A, h \in H \rangle \subseteq \ker(N). \end{aligned}$$

Since every $f \in \ker(N)$ can be written as $f = \prod_{h \in H} (f(h)_*)^{-1} f(h)_*^h$, we have $\ker(N) = [M_1^H(A), H]$ and hence $\overline{M_1^H(A)} \cong M_1^H(A)/\ker(N) \cong A$. Therefore (4.2) gives $d(G) = d(A) + d(H)$. \square

Proposition 4.4. *Let $G = A \rtimes H$, where A and H are p -groups and A is abelian. Let $\overline{A} = A/A^p[A, H]$ and $\pi : A \rightarrow \overline{A}$ the natural map. Let M be a minimal subgroup of A for which $\pi(M) = \overline{A}$. Then there is a rank preserving epimorphism*

$$\psi_2 : M \wr H \rightarrow G.$$

Proof. Let ψ_2 be the epimorphism from $A \wr H$ to G (see [2, Lemma 16.4.3]) defined for all $f \in M_1^H(A)$ and $h \in H$ by¹:

$$\psi_2(f, h) = \left(\prod_{\sigma \in H} f(\sigma)^\sigma, h \right).$$

We claim that the restriction of ψ_2 to $M \wr H$ remains surjective. As $\psi_2(1, h) = h$ for all $h \in H$, we have $H \subseteq \text{Im}(\psi_2)$. Since for every $a \in M$, $\psi_2(a_*, 1) = a$, we have

¹In [2], ψ_2 was defined by $\psi_2(f, h) = \left(\prod_{\sigma \in H} f(\sigma)^{\sigma^{-1}}, h \right)$. The source of the difference is in the definition of f^τ . In [2], $f^\tau(\sigma) = f(\tau\sigma)$.

$M \subseteq \text{Im}(\psi_2)$. As $\pi(M) = \bar{A}$ we have $MA^p[A, H] = A$ and hence $A^p[A, H]\langle M, H \rangle = G$. Since $A^p[A, H]$ is contained in $\Phi(G)$ this implies $\langle M, H \rangle = G$. It follows that $\text{Im}(\psi_2) \supseteq G$ which proves the claim. It remains to show that $d(M \wr H) = d(G)$. By Lemma 4.3, $d(M \wr H) = d(M) + d(H)$. By Lemma 2.2.(3), $d(M) = d(\bar{A})$. By Theorem 3.3, $d(G) = d(\bar{A}) + d(H)$. It follows that:

$$d(M \wr H) = d(M) + d(H) = d(\bar{A}) + d(H) = d(G).$$

□

Step III. Let G be a semiabelian p -group. The composition $\psi_2 \circ \psi_1$ gives a rank preserving epimorphism from $M \wr H \rightarrow G$ where $M \leq G$ is abelian and $H < G$ is semiabelian. To prove Theorem 1.3 we iterate this process using the following lemma:

Lemma 4.5. *Let A be a finite abelian group and $\psi : G \rightarrow \Gamma$ an epimorphism of finite groups. Then there is an epimorphism $\tilde{\psi} : A \wr G \rightarrow A \wr \Gamma$.*

Proof. We shall treat the Γ -module $M_1^\Gamma(A)$ as a G -module via the map ψ . By the Frobenius Reciprocity Theorem any homomorphism $i : A \rightarrow M_1^\Gamma(A)$ induces a homomorphism of G -modules $i^* : M_1^G(A) \rightarrow M_1^\Gamma(A)$ defined by:

$$i^*(f) = \prod_{g \in G} i(f(g))^g.$$

Let $i : A \rightarrow M_1^\Gamma(A)$ be the homomorphism $i(a) = a_*$. Then

$$(4.3) \quad i^*(f)(\gamma) = \left(\prod_{g \in G} f(g)^{\psi(g)} \right) (\gamma) = \prod_{\{g \in G : \psi(g) = \gamma\}} f(g).$$

We claim that i^* is surjective. Let $f \in M_1^\Gamma(A)$. For every $\gamma \in \Gamma$, fix an element $g_\gamma \in G$ for which $\psi(g_\gamma) = \gamma$ and define $\tilde{f} = \prod_{\gamma \in \Gamma} f(\gamma)_*^{g_\gamma} \in M_1^G(A)$. In particular, $\tilde{f}(g) = 1$ for $g \notin \{g_\gamma | \gamma \in \Gamma\}$ and $\tilde{f}(g_\gamma) = f(\gamma)$. By (4.3), $i^*(\tilde{f})(\gamma) = \tilde{f}(g_\gamma) = f(\gamma)$ for all $\gamma \in \Gamma$. It follows that for every $f \in M_1^\Gamma(A)$, $i^*(\tilde{f}) = f$, proving the claim.

Since $i^* : M_1^G(A) \rightarrow M_1^\Gamma(A)$ is an epimorphism of G -modules, i^* induces an epimorphism $\psi'_2 : M_1^G(A) \rtimes G \rightarrow M_1^\Gamma(A) \rtimes G$. Since $\ker(\psi) \triangleleft G$ acts trivially on $M_1^\Gamma(A)$, $\ker(\psi)$ is a normal subgroup of $M_1^\Gamma(A) \rtimes G$. In particular ψ'_2 induces an epimorphism

$$\psi_2 : A \wr G = M_1^G(A) \rtimes G \rightarrow (M_1^\Gamma(A) \rtimes G) / \ker(\psi) \cong M_1^\Gamma(A) \rtimes \Gamma = A \wr \Gamma.$$

□

Proof of Theorem 1.3. We argue by induction on $|G|$. Let $G = AH$ be a minimal decomposition. By Corollary 4.1, there is a rank preserving epimorphism $\psi_1 : A \rtimes H \rightarrow G$. Let $\bar{A} = A/A^p[A, H]$, $\pi : A \rightarrow \bar{A}$ the natural map and A_1 a minimal subgroup of A for which $\pi(A_1) = \bar{A}$. By Proposition 4.4 there is a rank preserving epimorphism $\psi_2 : A_1 \wr H \rightarrow A \rtimes H$. Thus $\psi = \psi_2 \circ \psi_1 : A_1 \wr H \rightarrow G$ is a rank preserving epimorphism.

By the induction hypothesis there are abelian p -groups A_2, \dots, A_r and a rank preserving epimorphism $\phi_0 : A_2 \wr (A_3 \wr \dots \wr A_r) \rightarrow H$. By Lemma 4.5, ϕ_0 induces an

epimorphism $\phi : A_1 \wr (A_2 \wr \dots \wr A_r) \rightarrow A_1 \wr H$. Using the equality $d(A_2 \wr \dots \wr A_r) = d(H)$ and Lemma 4.3 we have:

$$d(A_1 \wr H) = d(A_1) + d(H) = d(A_1) + d(A_2 \wr \dots \wr A_r) = d(A_1 \wr \dots \wr A_r)$$

and hence ϕ is rank preserving. It follows that $\psi \circ \phi : A_1 \wr (A_2 \wr \dots \wr A_r) \rightarrow G$ is a rank preserving epimorphism. \square

As a corollary we have:

Corollary 4.6. *The families \mathcal{G}_p and \mathcal{S}_p are equal.*

Proof. The inclusion $\mathcal{G}_p \subseteq \mathcal{S}_p$ follows from [3]. For abelian p -groups A_1, \dots, A_r the iterated wreath product $A_1 \wr (A_2 \wr \dots \wr A_r)$ is in \mathcal{G}_p . By Theorem 1.3, every semiabelian group is a rank preserving epimorphism of such an iterated wreath product and hence in \mathcal{G}_p . Thus, $\mathcal{G}_p = \mathcal{S}_p$. \square

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