THE LOCAL DIMENSION OF A FINITE GROUP OVER A NUMBER FIELD

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ABSTRACT. Let G be a finite group and K a number field. We construct a G-extension E/F, with F of transcendence degree 2 over K, that specializes to all G-extensions of $K_{\mathfrak{p}}$, where $\mathfrak{p}$ runs over all but finitely many primes of K. If furthermore G has a generic extension over K, we show that the extension E/F has the Hilbert–Grunwald property. The transcendence degree of the extension is compared to the essential dimension of G over K, and its arithmetic analogue.

1. INTRODUCTION

The essential dimension $\mathrm{ed}_K(G)$ of a finite group G is a central notion in algebra which measures the complexity of all G-extensions of overfields of a given field K [4]. More precisely, $\mathrm{ed}_K(G)$ is the minimal integer $d \geq 0$ for which there exists a *G-extension* E/F, of a field F of transcendence degree d over K, such that every G-extension of L, for every field $L \supseteq K$, is a specialization of E/F. Here, a G-extension E/F is an extension of étale algebras with Galois group G, and specialization is the usual procedure of tensoring with a residue field, see §2.1.

The *(essential) parametric dimension* $\mathrm{pd}_K(G)$ is an analogous notion in arithmetic geometry which measures the complexity of all G-extensions of K itself. More precisely, $\mathrm{pd}_K(G)$ is the minimal integer $d \geq 0$ for which there exist finitely[1] many G-extensions E_i/F_i, $i = 1, \dots, r$, with F_i of transcendence degree d over K, such that every G-extension of K is a specialization of E_i/F_i for some i. In general, $\mathrm{pd}_K(G) \leq \mathrm{ed}_K(G)$, and $\mathrm{pd}_K(G) < \mathrm{ed}_K(G)$ is possible even over the rationals $K = \mathbb{Q}$, see §A. Over number fields K, the inequality $\mathrm{pd}_K(G) > 1$ is known for various classes of groups G, see [9, 22, 24, 20], but in general $\mathrm{pd}_K(G)$ remains a mysterious invariant.

In the case where K is a number field, the natural arithmetic approach to parametrizing G-extensions L/K is to first parametrize their completions $L_{\mathfrak{p}} := L \otimes_K K_{\mathfrak{p}}$, where $\mathfrak{p}$ runs over primes of K. It is conjectured[2] [6] that for every finitely many G-extensions $L^{(\mathfrak{p})}/K_{\mathfrak{p}}$, $\mathfrak{p} \in S$, there exists a G-extension L/K such that $L \otimes_K K_{\mathfrak{p}} \cong L^{(\mathfrak{p})}$

[1] The motivation behind allowing finitely many G-extensions as opposed to one comes from rational connectedness, see Remark A.2. Alternative definitions may require other properties from F_i. The notion can also be compared with the arithmetic [30] and generic dimensions [19, §8.5].

[2] This is a special case of the more general conjecture in [6] relevant to G-extensions, cf. [17, 14, 26].

for all $\mathfrak{p} \in S \setminus T$, where $T = T(G, K)$ is a finite set of "bad primes" depending only on G and K. In this setting, the set of G-extensions $L^{(\mathfrak{p})}/K_\mathfrak{p}, \mathfrak{p} \in S$ is called a *Grunwald problem*, L is called a *solution* to it, and the conjecture is known for many solvable groups G, see [18] and [29, Theorem 9.5.9]. The conjecture implies that finitely many G-extensions E_i/F_i, $i = 1, \dots, r$ which specialize to every G-extension of K, also specialize to every G-extension of $K_\mathfrak{p}$, for all but finitely many primes $\mathfrak{p}$ of K. This motivates defining the *(essential) local dimension* $\mathrm{ld}_K(G)$ of G over K as the minimal integer $d \geq 0$ for which there exist finitely many G-extensions E_i/F_i, $i = 1, \dots, r$, with F_i of transcendence degree d over K, such that every G-extension of $K_\mathfrak{p}$ is a specialization of some E_i/F_i, for all but finitely many primes $\mathfrak{p}$ of K. In particular, $\mathrm{pd}_K(G) \geq \mathrm{ld}_K(G)$ when the conjecture holds.

Grunwald problems are a topic of independent interest inspired by the inverse Galois problem (IGP) and the crossed product construction [1, §11]. To parametrize solutions to Grunwald problems, we say that extensions E_i/F_i, $i = 1, \dots, r$ have the *Hilbert-Grunwald property* if every Grunwald problem $L^{(\mathfrak{p})}/K_\mathfrak{p}, \mathfrak{p} \in S$ for G, with S disjoint from a finite set of primes $T = T(E_1/F_1, \dots, E_r/F_r)$, has a solution within the set of specializations of E_i/F_i, $i = 1, \dots, r$. In this context, the quantity of interest is the *Hilbert–Grunwald dimension* $\mathrm{hgd}_K(G)$ of G over K, that is, the minimal integer $d \geq 0$ for which there exist finitely many G-extensions E_i/F_i, $i = 1, \dots, r$, with F_i of transcendence degree d over K, having the Hilbert–Grunwald property. Clearly, $\mathrm{hgd}_K(G) \geq \mathrm{ld}_K(G)$, and equality is possible in general, see Remark 4.3.

The Hilbert–Grunwald property was first studied by Dèbes–Ghazi [10] for a single K-regular G-extension $E/K(t)$ and Grunwald problems consisting of unramified extensions. Considering ramified extensions, [24] shows that $\mathrm{ld}_K(G) > 1$ and hence $\mathrm{hgd}_K(G) > 1$ for every group G containing a noncyclic abelian subgroup [24]. The notion of Hilbert–Grunwald dimension was first alluded to in [20, §4], which gave examples of (elementary abelian) groups with $\mathrm{hgd}(G) \leq 2$ and arbitrarily large essential dimension, giving rise to the question whether or not this invariant could ever be larger than 2.

We show that in fact $\mathrm{ld}_K(G) \leq 2$ holds in general, and $\mathrm{hgd}_K(G) \leq 2$ holds if G has a generic extension over K:

Theorem 1.1. *Let K be a number field and G be a finite group.*
(1) There exists a G-extension E/F, with F of transcendence degree 2 over K, such that every G-extension of $K_\mathfrak{p}$ is a specialization of E/F, for every prime $\mathfrak{p}$ of K outside a finite set T.
(2) If G admits a generic extension over K, then (a) F can be chosen purely transcendental over K, and (b) every Grunwald problem $L^{(\mathfrak{p})}/K_\mathfrak{p}$, $\mathfrak{p} \in S \setminus T$ has a solution within the set of specializations of E/F.

We note that although the assumption of local global principles (LGP) for certain varieties would imply $\mathrm{pd}_K(G) = \mathrm{ld}_K(G) = 2$ (see Remark A.3), we expect $\mathrm{pd}_K(G)$ to be arbitrarily large as G varies over finite groups. Theorem 1.1 and this expectation give a quantitative meaning to the assertion that the "local complexity" of the Galois theory of a given finite group is in general much less than the corresponding "global complexity". However, very little is known about groups G with $\mathrm{pd}_K(G) > 2$.

The heart of the proof of Theorem 1.1, given in Section 3, constructs extensions, of transcendence degree 2 over K, that specialize to all tamely ramified local extensions with prescribed Galois group D and inertia group I. Previous works [2, 24] give necessary conditions on such extensions depending on the local behavior at branch points. Here we give sufficient conditions on such extensions to maintain this local behavior under specialization, cf. Proposition 3.7. These conditions are achieved via the construction of transcendence degree 1 extensions that specialize to all unramified local extensions with group D/I and embedding these into suitable transcendence degree 2-extensions, cf. Corollary 3.4.

The constructed extensions E/F in fact have the stronger property of specializing to all G-extensions of $K'_{\mathfrak{p}'}$ for all but finitely many primes $\mathfrak{p}'$ of any finite extension K'/K. Thus, they relate to the notion of "arithmetic dimension" introduced by Mazur and O'Neil, see Remark 4.6.

The assumed existence of a generic extension for G over K in part (2) is equivalent to the retract rationality of the variety $X := \mathbb{A}^n/G$, for $G \leq S_n$ [19, Theorem 5.2.3]. This property is known to hold for many groups, cf. [19], and most classically for $G = S_n$. In fact, in the proof of Theorem 1.1 this assumption on X can be relaxed to a (two dimensional) weak approximation property, see Remark 4.4. Moreover, upon replacing the single G-extension E/F in Theorem 1.1 with finitely many G-extensions E_i/F_i, $i = 1, \ldots, r$, we expect that this weak approximation property would hold for many more groups G, see Remark 4.8.

Finally, we note that the notion of local dimension can be defined analogously for others fields K equipped with (infinite) sets of valuations. The specialization methods developed in Section 3.2, and the more basic Section 2.4, are expected to be applicable over various other fields K. It would be interesting to see how invariants such as $\mathrm{pd}_K(G)$ or $\mathrm{ld}_K(G)$ change as K varies, cf. Remark B.5.

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2. Preparations

2.1. **Notation.** Let K be a field of characteristic 0. Let $\overline{K}$ denote the algebraic closure of K, let μ_e the e-th roots of unity in $\overline{K}$, and $G_K := \mathrm{Gal}(\overline{K}/K)$ the absolute Galois group of K. In Section 1, the notion of a G-extension refers to an étale algebra L that is Galois over K with Galois group G. Such an algebra decomposes as $L = L_1 \oplus \cdots \oplus L_r$ for mutually K-isomorphic fields $L_1, \ldots, L_r$, which are Galois over K with Galois group $H \leq G$ (of index r). Call L_1 the *field underlying* L.

In the case where K is the fraction field of a Dedekind domain R and $\mathfrak{p}$ is a prime ideal of R, we denote by $K_{\mathfrak{p}}$ the completion of K at $\mathfrak{p}$. For a G-extension $L = L_1 \oplus \cdots \oplus L_r/K$ of étale algebras and a prime $\mathfrak{p}$ of K unramified in L/K, the residue extension $(L \otimes_R (R/\mathfrak{p}))/(R/\mathfrak{p})$ of L/K at $\mathfrak{p}$ is again a G-extension of étale algebras. We call the field underlying $L \otimes_R (R/\mathfrak{p})$ the *residue field* of L at $\mathfrak{p}$. Henceforth, we consider only G-extensions of fields.

Morphisms and function fields. Let F be a function field in finitely many variables with constant field K, and let $\varphi : G_F \to G$ be an epimorphism. The fixed field E of $\ker\varphi$ is then Galois over F with Galois group G. We may then choose a morphism $f : X \to Y$ of (smooth quasiprojective irreducible) varieties over K such that E and F are the function fields of X and Y, respectively. Note that Y is absolutely irreducible, whereas X is irreducible over K, but not necessarily absolutely irreducible. For short, call such f a *Galois cover* with group G. The epimorphism φ factors through an epimorphism $\varphi' : \pi_1(Y')_K \to G$ from the K-fundamental group of $Y' := Y \setminus D$ (where $D \subset Y$ is the branch divisor of f). One says that φ (or φ') *represents* f.

The *morphism of constants* φ_K of φ (resp. φ') is the natural projection $\varphi_K : G_K \to G_K/N$, where N is the image of $\ker\varphi$ under the natural projection $G_F \to G_K$ (resp. $\pi_1(Y')_K \to G_K$). The fixed field of $\ker\varphi_K$ is then the *constant field* of the fixed field E of $\ker\varphi$. We shall say that φ (or equivalently, E/F, or f) is *K-regular* if $\ker\varphi_K = G_K$ (or equivalently, if $\varphi(\pi_1(Y')_{\overline{K}}) = G$). This condition ensures that $\ker\varphi \cdot \pi_1(Y')_{\overline{K}} = \pi_1(Y')_K$, and hence $E = \overline{F}^{\ker\varphi}$ is linearly disjoint from $\overline{K}$.

Specialization. The residue field extension E_ν/K of E/F at a degree 1 place ν of F, which is unramified in E, is called the *specialization* of E/F at ν. From the varieties point of view, every K-rational point $t_0 \in Y(K)$, away from the branch locus of f, induces a section $s_{t_0} : G_K \to \pi_1(Y')_K$ to the projection $\pi_1(Y')_K \to G_K$, and thus a *specialization morphism* $\varphi_{t_0} = \varphi \circ s_{t_0} : G_K \to G$ (well defined up to conjugation). The two notions of specialization correspond to each other in the sense that E_{t_0} is the fixed field of the kernel of φ_{t_0}, where ν is the place corresponding to $t_0 \in Y(K)$.

The key case for us is the one where $F = K(t)$ is a rational function field. For any $t_0 \in \overline{K}$, denote by $t \mapsto t_0$ the place corresponding to the prime $(m_{t_0}) \lhd K[t]$, where

m_{t_0} is the minimal polynomial of t_0 over K. The specialization $E(t_0)_{t_0}/K(t_0)$ is then nothing but the residue extension of (the integral closure of $K[t]$ in) E at any prime $\mathfrak{Q}$ extending (m_{t_0}) in E. We also write $E_{t_0}/K(t_0)$ instead of $E(t_0)_{t_0}/K(t_0)$. In a similar vein, when K is the fraction field of a Dedekind domain with prime ideal $\mathfrak{p}$, we shall also write $E_{t_0}/K_{\mathfrak{p}}$ for the specialization of $EK_{\mathfrak{p}}/K_{\mathfrak{p}}(t)$ at $t \mapsto t_0 \in K_{\mathfrak{p}}$ (and refer to it as the specialization of $E/K(t)$ at $t \mapsto t_0 \in K_{\mathfrak{p}}$).

2.2. **ID pairs.** Let K be the fraction field of a Dedekind domain R of characteristic 0. A group D and a subgroup I are called an *ID (inertia-decomposition) pair over* K if D appears as a Galois group of a tamely ramified field extension $T_{\mathfrak{p}}/K_{\mathfrak{p}}$ with inertia group I for some prime $\mathfrak{p}$ of R. Note that as part of an ID pair, the embedding $I \to D$ is also fixed. The pair is called split if the projection $D \to D/I$ splits. For an ID pair $\pi = (I, D)$, let $P_K(\pi)$ denote the set of primes $\mathfrak{p}$ for which there exists a tamely ramified field extension $T_{\mathfrak{p}}/K_{\mathfrak{p}}$ with Galois group D and inertia group I. It is well known [16, Theorem 16.1.1] that $I \lhd D$, and that D/I is the Galois group of the residue extension at $\mathfrak{p}$.

The following lemma summarizes basic properties of (I, D) pairs. Let $C_D(I)$ denote the centralizer of I in D. Let $\sigma_{q,e} \in \mathrm{Aut}(K(\mu_e))$ denote the automorphism given by $\sigma_{q,e}(\zeta) = \zeta^q$ for all $\zeta \in \mu_e$. For a prime $\mathfrak{p}$ of K with finite residue field denote by $N(\mathfrak{p}) := |R/\mathfrak{p}|$ its norm.

Lemma 2.1. *Let $\pi = (I, D)$ be an ID pair over a fraction field K of a Dedekind domain R of characteristic 0. Then:*

(1) *There exists a (unique) homomorphism $\eta_\pi : D \to \mathrm{Gal}(K(\mu_e)/K)$ such that $I \cong \mu_e$ as D-modules, where $e = |I|$, and I is a D-module via conjugation while μ_e is a D-module via η_π. In particular, $\ker(\eta_\pi) = C_D(I)$.*

(2) *Assuming further that K is a number field, D/I and $\mathrm{Im}(\eta_\pi)$ are cyclic. Moreover, a prime $\mathfrak{p}$ of K with norm q is in $P_K(\pi)$ if and only if $\langle \sigma_{q,e} \rangle = \mathrm{Im}(\eta_\pi)$.*

Proof. Let $\mathfrak{p} \lhd R$ be a prime and $T_{\mathfrak{p}}/K_{\mathfrak{p}}$ a tamely ramified extension with Galois group D and inertia group I.

(1) Since $T_{\mathfrak{p}}/K_{\mathfrak{p}}$ is tamely ramified, [16, Prop. 6.2.1 and Cor. 16.2.7.(c)] show that $\mu_e \subseteq \overline{T}_{\mathfrak{p}}^I$ and that there is an isomorphism $\iota : I \to \mu_e$ of $\mathrm{Gal}(T_{\mathfrak{p}}^I/K_{\mathfrak{p}})$-modules, where I is a module over $\mathrm{Gal}(T_{\mathfrak{p}}^I/K_{\mathfrak{p}}) = D/I$ via conjugation in D. In particular, I is cyclic. Identifying I and μ_e via ι, we obtain a homomorphism $D \to \mathrm{Aut}(I) = \mathrm{Aut}(\mu_e)$ via the conjugation action in D. Let $\eta_\pi : D \to \mathrm{Aut}(K(\mu_e))$ be its composition with the natural inclusion $\mathrm{Aut}(\mu_e) \to \mathrm{Aut}(K(\mu_e))$, and let $V_\pi := \mathrm{Im}(\eta_\pi)$ be its image. Furthermore, $V_\pi \leq \mathrm{Gal}(K(\mu_e)/K)$ since $D/I = \mathrm{Gal}(T_{\mathfrak{p}}^I/K_{\mathfrak{p}})$ fixes $K_{\mathfrak{p}}$ and hence K. Since η_π sends every element $d \in D$, that satisfies $\tau^d = \tau^q$ for all $\tau \in I$, to $\sigma_{q,e} \in \mathrm{Gal}(K(\mu_e)/K)$, the map η_π is independent of the choice of ι. Hence η_π is

the unique map with the required property. Note that the kernel of the conjugation action on $I = \mu_e$ is $C_D(I)$, so that $\ker(\eta_\pi) = C_D(I)$.
(2) Since $R/\mathfrak{p}$ is finite, the Galois group D/I of the residue extension is cyclic, generated by the Frobenius element $\sigma \in D/I$. As $I \leq C_D(I)$, we have $\mathrm{Im}(\eta_\pi) \cong D/C_D(I)$ is cyclic.
"Only if part": Without loss of generality we may take $\mathfrak{p}$ and $T_\mathfrak{p}/K_\mathfrak{p}$ to be as above. Let $\sigma \in D$ be a lift of the Frobenius element $\sigma \in \mathrm{Gal}(T_\mathfrak{p}^I/K_\mathfrak{p}) = D/I$. Since $\langle\sigma\rangle = D/I$ and σ acts on μ_e by raising to the power of $q = N(\mathfrak{p})$, the element $\eta_\pi(\sigma) = \sigma_{q,e}$ generates V_π.
"If part": Let $K_\mathfrak{q}^{tr}$ be the maximal tamely ramified extension of $K_\mathfrak{q}$. By Iwasawa's theorem [29, Theorem 7.5.3], its Galois group $G_\mathfrak{q}^{tr} = \mathrm{Gal}(K_\mathfrak{q}^{tr}/K_\mathfrak{q})$ is isomorphic to the semidirect product $\langle\tau_\mathfrak{q}\rangle \rtimes \langle\sigma_\mathfrak{q}\rangle$ where $\langle\tau_\mathfrak{q}\rangle$ is the (procyclic) inertia group, $\langle\sigma_\mathfrak{q}\rangle \cong \hat{\mathbb{Z}}$ and $\tau_\mathfrak{q}^{\sigma_\mathfrak{q}} = \tau_\mathfrak{q}^q$. Since $\langle\sigma_{q,e}\rangle = V_\pi$, there exists $d \in \eta_\pi^{-1}(\sigma_{q,e})$ such that $\langle dI\rangle = D/I$. Therefore the epimorphism $\phi_\mathfrak{q} : G_\mathfrak{q}^{tr} \to D$, given by $\sigma_\mathfrak{q} \to d$ and $\tau_\mathfrak{q} \to \tau$ for a generator τ of I, is well defined, is an epirmophism and maps the inertia group to I. □

2.3. **Embedding problems.** We describe the relevant facts from [29, III.5]. A finite embedding problem over a field K is a pair $(\varphi : G_K \to G, \varepsilon : E \to G)$, where φ is a continuous epimorphism, and ε is an epimorphism of finite groups. A continuous homomorphism $\psi : G_K \to E$ is called a *solution* to (φ, ε) if the composition $\varepsilon \circ \psi = \varphi$. A solution ψ is called a *proper solution* if it is surjective. Two solutions ψ_1, ψ_2 to (φ, ε) are called equivalent if there exists $e \in E$ such that $\psi_1(\sigma) = e^{-1}\psi_2(\sigma)e$ for all $\sigma \in G_K$. A solution ψ to (φ, ε) over $L \supseteq K$ is a solution to (φ_L, ε), where $\varphi_L : G_L \to G$ is an extension of $\varphi_{|\,\mathrm{Gal}(\overline{K}\cdot L/L)}$ to G_L. In case $L/K(t)$ is K-regular and K_1/K is a finite extension, $\psi : G_L \to E$ is called K_1-regular if K_1 is the fixed field of $\ker \psi_K$. In this case, K_1/K is the constant extension of $\overline{L}^{\ker\psi}/L$.

If (φ, ε) has a proper solution ψ (over K), then we may identify E (resp. G) with $\mathrm{Gal}(M/K)$ (resp. $\mathrm{Gal}(L/K)$), where M and L are the fixed fields of $\ker\psi$ and $\ker\varphi$, respectively. Then ε becomes the restriction map $\mathrm{Gal}(M/K) \to \mathrm{Gal}(L/K)$.

If $A := \ker\varepsilon$ is abelian, then G acts on A via conjugation in E, and hence G_K acts on A through φ. If further (φ, ε) has a solution ψ, Hoechsmann's theorem implies that the set of equivalence classes of solutions to (φ, ε) is in bijection with classes in $H^1(G_K, A)$. More specifically, the bijection attaches to a class $[\chi]$ of $\chi \in Z^1(G_K, A)$ the solution $\chi \cdot \psi : G_K \to A$, $\sigma \mapsto \chi(\sigma)\psi(\sigma)$.

The embedding problem (φ, ε) is called *Brauer* if the kernel A is isomorphic to μ_n as a G_K-module for $n \geq 2$. In particular, $A \cong \mu_n$ is fixed by $\ker\varphi$. A general reference on Brauer embedding problems is [27, Chapter IV,§7]. By Kummer theory, $H^1(G_K, \mu_n) \cong K^\times/(K^\times)^n$. More explicitely, letting M and L be as above, we have

$M = L(\sqrt[n]{a})$ for $a \in L^\times$ and the field fixed by $\ker(\chi \cdot \psi)$ is $L(\sqrt[n]{ab})$ for $b \in K^\times/(K^\times)^n$ corresponding to $[\chi]$.

2.4. **Relating arithmetic and geometric decomposition groups.** In this section K is the fraction field of a Dedekind domain R of characteristic zero such that its residue field at every place is perfect. We describe the decomposition group of a specialization E_{t_0}/K at a prime $\mathfrak{p}'$, when $t_0 \in K$ is $\mathfrak{p}'$-adically close to a branch point t_1 of a K-regular G-extension $E/K(t)$, giving a more explicit version of [24].

Let D_{t_1} (resp., I_{t_1}) be the decomposition group (resp., the inertia group) of $E(t_1)/K(t_1,t)$ at the prime $(t-t_1)\,K[t-t_1]$, and let $D_{t_1,\mathfrak{p}'}$ be the decomposition group of $E(t_1)_{t_1}/K(t_1)$ at a prime $\mathfrak{p}'$ of $K(t_1)$, so that $D_{t_1,\mathfrak{p}'}$ is canonically identified with a subgroup of D_{t_1}/I_{t_1}. Let $\varphi : D_{t_1} \to D_{t_1}/I_{t_1}$ be the natural projection.

For a prime $\mathfrak{p}$ of R, denote by $v_\mathfrak{p} : K \to \mathbb{Z}$ the associated valuation. If $\mathfrak{p}$ is not in a finite set of bad places $\mathcal{S}_1$ and t_1 is $\mathfrak{p}$-integral, [24, Lemma 3.2] associates to it a unique degree 1 prime $\mathfrak{p}' = \mathfrak{p}'(t_0, t_1, \mathfrak{p})$ extending $\mathfrak{p}$ to $K(t_1)$ such that $v_{\mathfrak{p}'}(t_0 - t_1) > 0$, that is, t_0 and t_1 meet at $\mathfrak{p}'$.

For a separable monic polynomial $P(t,x) \in R[t][x]$, we will denote by $\Delta_x(t) \in R[t]$ the reduced discriminant of P with respect to x, that is, the radical of the discriminant of P with respect to x. In particular, Δ_x is also separable. Let $\mathcal{S}_R(P)$ be the finite set of primes $\mathfrak{p}$ of R for which $\Delta_x(t) \bmod \mathfrak{p}$ is either inseparable or of smaller degree than $\deg_t \Delta_x(t)$.

Theorem 2.2. *Let $P(t,x)$ be a separable monic polynomial over $R[t]$ with splitting field E such that $E/K(t)$ is a G-extension. Denote by $\mathcal{S}_0$ $(= \mathcal{S}_0(E/K(t)))$ the union of the set of primes of R dividing $|G|$ with the finite set $\mathcal{S}_R(P)$. Suppose $\mathfrak{p}$ is a prime of R not in $\mathcal{S}_0$, $t_0 \in \mathbb{P}^1(K)$ is not a branch point of $E/K(t)$ while t_1 is a finite branch point. Suppose $v_{\mathfrak{p}'}(t_0 - t_1)$ is positive and coprime to $|I_{t_1}|$. Then:*

1. *The decomposition group $D_{t_0,\mathfrak{p}}$ is conjugate in G to $\varphi^{-1}(D_{t_1,\mathfrak{p}'})$.*
2. *The unramified part of the completion $E_{t_0} \cdot K_\mathfrak{p}$ of E_{t_0} at $\mathfrak{p}$ is $(E(t_1))_{t_1} \cdot k(t_1)_{\mathfrak{p}'}$.*

Remark 2.3. The same conclusion is shown in [24, Theorem 4.1] under the assumption that $\mathfrak{p}$ is not in a set $\mathcal{S}_{\text{exc}}$ of exceptional places which is described as the union of the set $\mathcal{S}_{\text{bad}}$ of bad primes from the Inertia specialization theorem [2, 25], and four other sets $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4$. The set $\mathcal{S}_{\text{bad}}$ consists of 1) the set $\mathcal{S}_{\text{bad},1}$ of primes $\mathfrak{p}$ of K where two branch points meet; 2) the set $\mathcal{S}_{\text{bad},2}$ of primes $\mathfrak{p}$ with "vertical ramification", i.e., such that $\mathfrak{p}R[t]$ is ramified (in the sense of [2, Section 2]) in the integral closure of $R[t]$ in E; 3) the set $\mathcal{S}_{\text{bad},3}$ of primes $\mathfrak{p}$ such that at least one branch point of $E/K(t)$ is not $\mathfrak{p}$-integral; and 4) the set $\mathcal{S}_{\text{bad},4}$ of primes that divide the order of G.

The sets $\mathcal{S}_i$, $i = 1, \dots, 4$ are unions of sets $\mathcal{S}_i(t_1)$ where t_1 runs through the finite branch points of $E/F(t)$. The set $\mathcal{S}_1(t_1)$ consists of places where the minimal polynomial of t_1 is either not $\mathfrak{p}$-integral or nonseparable. The set $\mathcal{S}_3(t_1)$ is the set of places of K ramified in $K(t_1)$. The set $\mathcal{S}_4(t_1)$ is then the set of primes $\mathfrak{p}$ such that $v_{\mathfrak{p}'}(d - t_1) > 0$ for some root $d \neq t_1$ of Δ_x and some prime $\mathfrak{p}'$ of $K(t_1)$ over $\mathfrak{p}$.[3] Finally, the set $\mathcal{S}_2(t_1)$ is the set of primes $\mathfrak{p}$ for which the following holds (cf. [24, Lemma 4.3]): there exists some intermediate field M of $E \cdot E(t_1)_{t_1}/K(t)$ such that every primitive element y of $M/K(t)$ has a Puiseux expansion in $s := t - t_1$ in which some coefficient is of $\mathfrak{p}$-adic absolute value $|y|_{\mathfrak{p}} > 1$ (equivalently, of negative $\mathfrak{p}$-adic valuation).

Proof of Theorem 2.2. In view of Remark 2.3, this follows from [24, Theorem 4.1], once we show that for every $\mathfrak{p} \in \mathcal{S}_{exc}$ (as described above) coprime to $|G|$, either the reduction of Δ_x is nonseparable, or it is of smaller t-degree than $\Delta_x(t)$. The set of finite branch points of $E/K(t)$ is necessarily contained in the set of roots of $\Delta_x(t)$, and thus $\mathfrak{p} \in \mathcal{S}_{\text{bad},1}$ implies that $\Delta_x(t)$ mod $\mathfrak{p}$ is either inseparable (this happens when two finite branch points meet mod $\mathfrak{p}$) or of degree smaller than $\deg_t(\Delta_x(t))$ (this happens when ∞ and a finite branch point meet mod $\mathfrak{p}$). The set $\mathcal{S}_{\text{bad},2}$ is contained in the set of those primes modulo which $\Delta_x(t)$ is congruent to 0 (see, e.g., Addendum 1.4.c) of [10]). If $\mathfrak{p} \in \mathcal{S}_{\text{bad},3}$, then the leading coefficient of $\Delta_x(t)$ is divisible by $\mathfrak{p}$, i.e., again reduction mod $\mathfrak{p}$ reduces the degree of Δ_x. We have assumed $\mathfrak{p}$ is coprime to $|G|$, so that $\mathfrak{p} \notin \mathcal{S}_{\text{bad},4}$. The set $\mathcal{S}_4$ consists of primes $\mathfrak{p}$ for which $\Delta_x(t)$ mod $\mathfrak{p}$ is inseparable. If $\mathfrak{p} \in \mathcal{S}_3$, it ramifies in $K(t_1)/K$. Let $m_{t_1}(t)$ be the minimal polynomial of t_1 over K, so that $m_{t_1} \mid \Delta_x$. As $\mathfrak{p}$ ramifies in $K(t_1)$, it divides the discriminant of m_{t_1}, that is, m_{t_1} and hence Δ_x are inseparable modulo $\mathfrak{p}$. If $\mathfrak{p} \in \mathcal{S}_1$, then m_{t_1} and hence Δ_x are nonseparable or nonintegral at $\mathfrak{p}$.

Lastly, consider the set $\mathcal{S}_2$. If the roots of the given polynomial $P(t, x)$ have Puiseux expansions in $s := t - t_1$ all of whose coefficients are of $\mathfrak{p}$-adic absolute value ≤ 1, then it is elementary to construct primitive elements for each intermediate field of $E \cdot E(t_1)_{t_1}/K(t)$ with the same property, e.g. by taking a suitable polynomial combination of the roots of P with $\mathfrak{p}$-integral coefficients. It therefore suffices to show that, if $\mathfrak{p}$ is coprime to G and such that Δ_x mod $\mathfrak{p}$ is separable, then any root z of $P(t, x)$ has a Puiseux series expansion $z = z(s) = \sum_{n \geq 0} a_n s^{n/e}$ in s with coefficients a_n of p-adic absolute value ≤ 1.[4] This holds essentially due to results by Dwork and Robba on convergence of p-adic power series ([15]). Indeed, separability of Δ_x mod $\mathfrak{p}$ implies that every root d of $\Delta_x(t)$ other than $d = t_i$ satisfies $|d - t_i|_{\mathfrak{p}} \geq 1$. Since additionally $\mathfrak{p}$ is chosen coprime to $|G|$, [33, Proposition 4.1] (which extends slightly [15, Theorem 2.1]) implies that the power series $\sum_{n \geq 0} a_n y^n$ converges in the open disk of $\mathfrak{p}$-adic absolute value 1. By [3, Corollary 4.3] (and monicity of P), the claim $|a_n|_{\mathfrak{p}} \leq 1$ follows for all n. □

[3]Note that t_1 itself is also a root of Δ_x from the definition of a branch point.

[4]Note that the expansion of z contains no negative n due to integrality of z by the choice of P.

Next, we shall consider how the set $\mathcal{S}_R(P)$ varies when P runs through a family of polynomials, that is, $P \in R[V][t,x]$, where $R[V]$ denotes the coordinate ring of an affine variety V defined over R. Denote by $P_v \in T[t,x]$ its specialization at $v \in V(T)$ for an R-algebra T. For $v \in V(K)$, we write $v_\mathfrak{p} \in V(R/\mathfrak{p})$ to assert that its reduction mod $\mathfrak{p}$ is defined and denoted by $v_\mathfrak{p}$.

Lemma 2.4. *Let V be an irreducible affine variety over R, let t be an indeterminate over R, and $P \in R[V][t,x]$ a monic separable polynomial in x. Then there exists a proper (closed) subvariety V_P of V with the following property. For every $v \in V(K) \setminus V_P(K)$ and prime $\mathfrak{p} \lhd R$, we have $\mathfrak{p} \notin \mathcal{S}_R(P_v)$ if and only if $v_\mathfrak{p} \in V(R/\mathfrak{p}) \setminus V_P(R/\mathfrak{p})$.*

Proof. Let $\Delta = \Delta_x(P) \in R[V][t]$ denote the reduced discriminant of P. Let $\ell_t(\Delta) \in R[V]$ be its leading coefficient, and let $\Delta_t \in R[V]$ be the discriminant of Δ with respect to t. Define V_P to be the union of two proper[5] closed subvarieties: 1) $\Delta_t = 0$ and 2) $\ell_t(\Delta) = 0$. Letting U be either K or $R/\mathfrak{p}$, the specialization $(\Delta_t)_v = (\Delta_v)_t \in U$ is zero if and only if $\Delta_v \in U[t]$ is nonseparable for every $v \in V(U)$. Also note that $\ell_t(\Delta_v) = 0$ if and only if the t-degree of $\Delta_v \in U[t]$ is the same as that of $\Delta \in R[V][t]$. In particular, $\Delta_v \in K[t]$ is separable of the same t-degree as $\Delta \in R[V][t]$, for every $v \in V(K) \setminus V_P(K)$. Moreover, this shows that $\Delta_{v_\mathfrak{p}}$ is separable of t-degree equal to $\deg_t \Delta = \deg_t \Delta_v$ if and only if $v_\mathfrak{p} \notin V_P(R/\mathfrak{p})$. □

Remark 2.5. Assume $V = \mathbb{A}^1$, so that $R[V] = R[s]$ for an indeterminate s. Then V is of dimension 1, and hence there is a constant d_P such that $|V_P(R/\mathfrak{p})| \leq d_P$, for every prime $\mathfrak{p}$ of R. In particular for every prime $\mathfrak{p}$, there are at most d_P residue classes $s \equiv s_0 \bmod \mathfrak{p}$ for which $\mathfrak{p} \in \mathcal{S}_R(P_{s_0})$.

Finally, we note that although the residue extension is fixed as t_0 varies through values as in Theorem 2.2, the ramified part is quite arbitrary [24, Theorem 4.4]:

Theorem 2.6. *Let $E/K(t)$ be a K-regular Galois extension where K is the fraction field of a Dedekind domain R, and $t_1 \in \overline{K}$ a branch point. Let D, I, and $N/K(t_1)$ be the Galois group, inertia group, and residue extension, respectively, of the completion at $t \mapsto t_1$. Let $\mathfrak{p} \lhd R$ be a prime away from the finite set $\mathcal{S}_0$ of Theorem 2.2, with a degree 1 prime $\mathfrak{P}$ of $K(t_1)$ lying over it. Let $T_\mathfrak{p}/K_\mathfrak{p}$ be a D-extension with inertia group I such that $T_\mathfrak{p}^I \cong NK(t_1)_\mathfrak{P}$ ($\cong NK_\mathfrak{p}$). Then there exist infinitely many specializations E_{t_0}/K at $t_0 \in K$ whose completion at $\mathfrak{p}$ is $T_\mathfrak{p}/K_\mathfrak{p}$.*

Remark 2.7. Let $P \in R[t,x]$ be a polynomial with splitting field E whose roots are integral over $R[t]$. One may write the completion of E at $t \mapsto t_1$ as $N((\sqrt[e]{a(t-t_1)}))$ for $e = |I|$ and $a \in N^\times$. In [24], the exceptional set of primes for Theorem 2.6 is

[5] Note that the discriminant of a nonzero linear polynomial is defined to be the leading coefficient, hence the first variety is proper.

instead given as the set $\mathcal{S}'_{exc}$ consisting of the exceptional primes of Theorem 2.2 together with the finitely many primes $\mathfrak{p}$ where $e \nmid v_{\mathfrak{p}}(a)$. However we claim that the latter primes are contained in $\mathcal{S}_R(P)$. First, by possibly replacing a, we may assume $v_{\mathfrak{p}}(a) > 0$. Let α be a root of P that is not fixed by I, and $\beta = \alpha^i$ a nontrivial conjugate of α by some $i \in I$. Since α, β are integral over $R[t]$, they are integral over $R'[[t-t_1]]$ where R' is the integral closure of R in N. Let $\mathfrak{p}'$ be a prime of R' over $\mathfrak{p}$ which divides $u := \sqrt[e]{a(t-t_1)}$. Viewing α and β as elements of the integral closure of $R'[[t-t_1]]$ in $N((u))$, the difference $\alpha - \beta$ is divisible by u and hence by $\mathfrak{p}'$. It follows that the discriminant of P is zero mod $\mathfrak{p}'$ and hence mod $\mathfrak{p}$, proving the claim.

2.5. **Twists and specializations.** Let K be a field of characteristic 0, let $f : X \to Y$ be a Galois cover with group G of smooth projective irreducible varieties over K, with Y absolutely irreducible. Let $\varphi : \pi_1(Y')_K \to G$ be an epimorphism representing f (see Section 2.1). We are interested in whether a given morphism $\psi : G_K \to G$ occurs as a specialization morphism of the given φ. Following [11], this is equivalent to a question about K-rational points of a certain cover f^ψ, called the *twist* of f by ψ. Here, $f^\psi : \tilde{X} \to Y$ is defined as the cover represented by the following (not necessarily surjective) homomorphism $\varphi^\psi : \pi_1(Y')_K \to \tilde{G}$, where $\tilde{G} \leq Sym(G)$ denotes the image of G under the regular permutation action,

$$(\varphi^\psi(\theta))(x) := \varphi(\theta)x(\psi(\overline{\theta}))^{-1},\ \theta \in \pi_1(Y')_K, x \in G,$$

with $\overline{\theta}$ the image of θ under projection $\pi_1(Y')_K \to G_K$.

We only recall the properties of f^ψ relevant for us and refer to [11, Section 3] for more details. Firstly, after base change from K to the fixed field of $\ker \psi$ and possibly restricting to any connected component of the base-changed covers, $f^\psi : \tilde{X} \to Y$ becomes isomorphic to f. Moreover, by the so-called "twisting lemma" [10, Lemma 2.1], [11, Lemma 3.2], $\tilde{X}$ has a rational point at $\mathbf{t}_0 \in Y(K)$ (away from the branch locus) if and only if the specialization morphism of φ at $\mathbf{t}_0$ is ψ up to conjugation.

Note that a trivially necessary condition for $\tilde{X}$ to have a K-rational point is for f^ψ to be K-regular, i.e., have trivial morphism of constants, which requires the morphism of constants φ_K of φ to fulfill $\varphi_K = \theta \circ \psi$, where (up to conjugation) θ is the canonical epimorphism of G onto the image of φ_K. In this case, say for short that ψ *projects* onto φ_K.

For a number field K and a prime $\mathfrak{p}$ of K, denote by $\varphi_{\mathfrak{p}}$ and $G_{\mathfrak{p}}$ the restriction of φ to $\pi_1(Y')_{K_{\mathfrak{p}}}$ and its image, respectively. We can now state the following theorem on specializations of function field extensions $E/K(t)$ with a prescribed local behavior, which extends [10, Theorem 1.2] to cases where E is not necessarily K-regular.

Theorem 2.8. *Let $f : X \to \mathbb{P}^1$ be a Galois cover with group G, defined over K, represented by $\varphi : \pi_1((\mathbb{P}^1)')_K \to G$ and with function field extension $E/K(t)$. Then*

there exists a bound $L(\varphi)$ such that for every prime $\mathfrak{p}$ of K with norm $N(\mathfrak{p}) > L(\varphi)$ the following holds: for every unramified morphism $\psi : G_{K_\mathfrak{p}} \to G$ such that ψ projects onto the morphism of constants $\varphi_{K_\mathfrak{p}}$ of $\varphi_\mathfrak{p}$, there exist infinitely many $t_0 \in K$ such that the specialization of $\varphi_\mathfrak{p}$ at t_0 is ψ (up to conjugation).
In particular, for every unramified extension $F/K_\mathfrak{p}$ with Galois group embedding into G and which contains the full constant field of $EK_\mathfrak{p}/K_\mathfrak{p}(t)$, there exist infinitely many $t_0 \in K$ such that the completion of E_{t_0}/K at $\mathfrak{p}$ equals $F/K_\mathfrak{p}$.

Proof. Via the twisting lemma, it suffices to show that the twist f^ψ, viewed as a morphism over $K_\mathfrak{p}$, has infinitely many $K_\mathfrak{p}$-points. As shown in [11, Section 3.1.1], the assumption that ψ projects onto $\varphi_{K_\mathfrak{p}}$ implies that the twist f^ψ is $K_\mathfrak{p}$-regular, i.e., of the form $f^\psi : X_\mathfrak{p} \to \mathbb{P}^1_{K_\mathfrak{p}}$ with $X_\mathfrak{p}$ absolutely irreducible. Furthermore, $X_\mathfrak{p}$ is of the same genus as X, since the two become isomorphic after suitable base change. We may then follow the proof of [10, Theorem 1.2] from here on. The first assertion follows via considering first the mod-$\mathfrak{p}$ reduction $\overline{X_\mathfrak{p}}$ of $X_\mathfrak{p}$, application of the Lang-Weil bound for this reduction in order to ensure the existence of simple mod-$\mathfrak{p}$ points, and then Hensel lifting to ensure the existence of $K_\mathfrak{p}$-points. The assertion about fields follows by choosing a suitable epimorphism $G_{K_\mathfrak{p}} \to \mathrm{Gal}(F/K_\mathfrak{p})$, projecting onto the prescribed $\varphi_{K_\mathfrak{p}}$, which is possible due to the containment $EK_\mathfrak{p} \subseteq F$. □

Remark 2.9. An explicit application of the Lang-Weil bound in the above proof shows the following stronger statement: There are constants $a, b > 0$ depending only on $E/K(t)$ such that for at least $a \cdot N(\mathfrak{p}) - b$ residue classes r mod $\mathfrak{p}$, the condition $t_0 \equiv r$ mod $\mathfrak{p}$ is sufficient to yield the assertions of Theorem 2.8. Indeed, this is due to the fact that only a bounded number (precisely, $\leq |G|$) points of $\overline{X_\mathfrak{p}}$ lie over the same point of $\mathbb{P}^1$, and only a bounded number of points (namely the branch points) have to be excluded as specialization values t_0.

3. The construction and its specializations

We now proceed to the construction in Theorem 1.1. For an ID pair $\pi = (I, D)$ over a number field K, Section 3.1 constructs a D-extension with two parameters that contains μ_e for $e = |I|$, and is as regular as possible. Section 3.2 shows that the constructed extension specializes to every tamely ramified extension of $K_\mathfrak{p}$ with Galois group D and inertia group I, for all but finitely many primes $\mathfrak{p}$ of K.

3.1. **Construction.** The following two lemmas are used to construct the residue extensions of the desired extension:

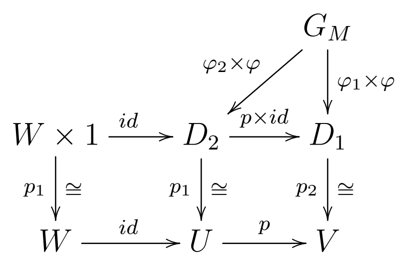
Lemma 3.1. *Let F_0 be a one variable function field*[6] *over a field K, let L/K be a cyclic extension, and $\varphi : G_{F_0} \to V$ the restriction map to $V := \mathrm{Gal}(LF_0/F_0)$. Let*

[6] The proof requires merely that F_0 is a function field over K admitting a K-regular U-extension.

U be a cyclic group with an epimorphism $p : U \to V$. Then (φ, p) has an L-regular proper solution over M for some K-regular extension M/F_0.

Proof. Set $W := \ker p$. Since U is cyclic, there exists a K-regular epimorphism $\varphi_2 : G_{F_0} \to U$. Let $\varphi_1 : G_{F_0} \to V$ be its composition with p. Since φ_1 is K-regular, $\ker \varphi_1 \cdot G_{\overline{K}F_0} = G_{F_0}$ and hence $\ker \varphi_1 \cdot \ker \varphi = G_{F_0}$. Thus, the map $\varphi_1 \times \varphi : G_{F_0} \to V \times V$, $\sigma \mapsto (\varphi_1(\sigma), \varphi(\sigma))$, is onto. Choose M to be the subfield fixed by $(\varphi_1 \times \varphi)^{-1}(D_1)$ where $D_1 \leq V \times V$ is the diagonal subgroup.

Since the image of the map $\varphi_2 \times \varphi : G_M \to U \times V$ is the diagonal subgroup $D_2 := \{(u, p(u) \mid u \in U\}$, the projection $p_1 : U \times V \to U$ maps D_2 isomorphically to U, yielding an epimorphism $\psi := p_1 \circ (\varphi_2 \times \varphi) : G_M \to U$. Letting $p_2 : U \times V \to V$ be the projection to V, since $p \circ p_1 = p_2 \circ (p \times id)$, and $p \circ \varphi_2 = \varphi_1$, we have $p \circ \psi = p_2 \circ (p \times id) \circ (\varphi_2 \times \varphi) = \varphi$. Hence ψ is a (proper) solution.



It remains to show that ψ is L-regular. First note that $\psi^{-1}(W) = \ker \varphi_{|G_M} = G_{LM}$, and LM/M is our constant extension. Thus to prove L-regularity, it suffices to show $\psi(G_{\overline{K}M}) \supseteq W$. We claim that $G_{\overline{K}M} \supseteq G_{\overline{K}F_0} \cap \varphi_2^{-1}(W)$. Since $\varphi_2(G_{\overline{K}F_0}) = U$ as φ_2 is K-regular, the claim shows that $\varphi_2(G_{\overline{K}M}) = W$, and hence $\psi(G_{\overline{K}M}) = p_1 \circ (\varphi_2 \times \varphi)(G_{\overline{K}M}) \supseteq W$, as desired for L-regularity. To prove the claim, recall that M is the fixed field of $(\varphi_2 \times \varphi)^{-1}(D_2)$, which contains $(\varphi_2 \times \varphi)^{-1}(W \times 1) = \varphi_2^{-1}(W) \cap \ker \varphi$. Thus, $G_{\overline{K}M} = G_{\overline{K}F_0} \cap (\varphi_2 \times \varphi)^{-1}(D_2) \supseteq G_{\overline{K}F_0} \cap \varphi_2^{-1}(W) \cap \ker \varphi$. Since $\ker \varphi = G_{LF_0} \supseteq G_{\overline{K}F_0}$, this gives $G_{\overline{K}M} \supseteq G_{\overline{K}F_0} \cap \varphi_2^{-1}(W)$, proving the claim. □

Lemma 3.2. *In the setup of Lemma 3.1, assume further that V is a p-group for a prime $p \neq \operatorname{char} K$, where either p is on odd prime or $\mu_4 \subseteq K$, and that $L = K(\mu_e)$ for $e = p^s$. Then (φ, p) has an L-regular proper solution over F_0.*

Proof. Set $W := \ker p$ and assume[7] s is maximal such that $L = K(\mu_{p^s})$. Since $L = K(\mu_{p^s})$, the embedding problem (φ, p) has a solution $\psi_0 : G_K \to U$ given by restriction to $\mathrm{Gal}(K(\mu_{p^{s+k}})/K)$, where $p^k = |W|$. Extend ψ_0 to G_{F_0} by composing with the restriction $G_{F_0} \to G_K$. Let $\varphi_2 : G_{F_0} \to W$ be a K-regular epimorphism, that is, $\varphi_2(G_{\overline{K}F_0}) = V$. Since V is abelian, $\psi = \psi_0 \cdot \varphi_2$ is also a solution to (φ, p).

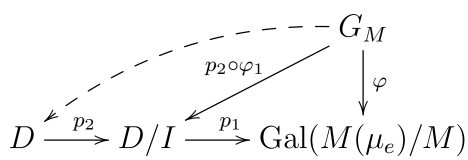
[7]This assumption regards only the case where L/K is the trivial extension.

Since U is cyclic of p-power order, a preimage under p of a generator of V is a generator of U, and hence ψ is proper.

Since $p \circ \psi = p \circ \psi_0$, the fixed field of $\ker(p \circ \psi)$ is $F_1 := F_0(\mu_e)$. Let $\psi_W : G_{F_1} \to W$ (resp. $\varphi_1 : G_{F_1} \to W$) be the restriction of ψ (resp. φ_2) to G_{F_1}. Since $\varphi_2(G_{\overline{K}F_0}) = V$ as φ_2 is K-regular, and since $\psi_0(G_{\overline{K}F_0}) = 1$, we have $\varphi_1(G_{\overline{K}F_0}) = W$, that is, φ_1 is L-regular. As $\psi_0(G_{\overline{K}F_0}) = 1$, we have $\psi(G_{\overline{K}F_0}) = \varphi_1(G_{\overline{K}F_0}) = W$, so that ψ is L-regular. □

The following lemma constructs the desired extension from the residue extension:

Lemma 3.3. *Let D be a finite group, $I \lhd D$ a cyclic normal subgroup of order e, and $p_2 : D \to D/I$ the natural projection. Suppose F_π is a complete discrete valuation field with characteristic 0 residue field M. Assume that $I \cong \mu_e$ as D-modules, where D acts on I by conjugation, and on μ_e via a surjection $p_1 : D \to \mathrm{Gal}(M(\mu_e)/M)$. Finally suppose $\varphi_1 : G_M \to D$ is a solution to $(\varphi, p_1 \circ p_2)$, where φ is the restriction map. Then $(p_2 \circ \varphi_1, p_2)$ has a proper totally ramified solution over F_π.*



Proof. Embedding the algebraic closure $\overline{M}$ of M into that $\overline{F}_\pi$ of F_π, we extend φ and φ_1 to G_{F_π}. Since $\ker p_2 = I \cong \mu_e$ as G_{F_π}-modules the equivalence classes to solutions to (φ, π) are in one to one correspondence with elements of $H^1(G_{F_\pi}, \mu_e)$, as described in Section 2.3. Moreover, letting N be the fixed field of $\ker(p_2 \circ \varphi_1)$, the fixed field of $\ker \varphi_1$ is of the form $N(\sqrt[e]{b})$ for $b \in N^\times$. Pick $s_\pi \in F_\pi^\times$ to be a uniformizer, and let $\alpha_\pi \in Z^1(G_{F_\pi}, \mu_e)$ be a class corresponding to $s_\pi (F_\pi^\times)^e$ via the Kummer isomorphism $H^1(G_{F_\pi}, \mu_e) \cong F_\pi^\times/(F_\pi^\times)^e$. As in Section 2.3, the fixed field of $\ker(\alpha_\pi \cdot \varphi_1)$ is $N(\sqrt[e]{bs_\pi})$. Thus $\alpha_\pi \cdot \varphi_1$ is proper totally ramified solution. □

Combining the above lemmas we obtain the following constructions. Recall that Lemma 2.1 associates to every ID pair (I, D) with $e := |I|$, a unique map $\eta_\pi : D \to \mathrm{Gal}(K(\mu_e)/K)$ with kernel $C_D(I)$, whose image is denoted by V_π.

Corollary 3.4. *Let $\pi = (I, D)$ be an ID pair over a number field K, and $\eta_\pi : D \to V_\pi$ as above. Let t be transcendental over K, and $F_0 = K(t)$. Then there exist infinitely many places $s \mapsto s_\pi$ of $F_0(s)$ such that the completion F_π at $s \mapsto s_\pi$ admits a D-extension E_π/F_π with inertia group I and residue extension N/M satisfying:*

(a) *M/F_0 is a finite and M is K-regular;*
(b) *the constant field of N is $K(\mu_e)$;*

(c) $N^{C_D(I)} \cong M(\mu_e)$, *where* $C_D(I)$ *acts via the natural quotient* $D \to D/I$.

Proof. Let U be a cyclic subgroup of D which projects onto D/I under the projection $p_I : D \to D/I$ modulo I. In particular, the action of U on I by conjugation factors through that of D/I. Set $K_0 := K(\mu_e)^{V_\pi}$, so that $K_0(\mu_e)/K_0$ is cyclic with Galois group V_π. By Lemma 3.1, there exists a finite K_0-regular extension M/F_0 (so that (a) holds), and a proper $K_0(\mu_e)$-regular solution $\varphi_1 : G_M \to U$ to $(\eta_\pi : U \to V_\pi, \varphi)$ over M, where $\varphi : G_{F_0} \to V_\pi$ is the restriction map.

Let $\hat{D} = I \rtimes U$ with semidirect product action given by conjugation in D, and let $\hat{p}_I : \hat{D} \to U$ be the projection modulo I. Let s_π be a primitive element for M/F_0, and F_π the completion of $F_0(s)$ at $s \to s_\pi$. Since $(\hat{p}_I, \varphi_1)$ has a trivial solution φ_1, it also has a proper totally ramified solution $\hat{\psi} : G_{F_\pi} \to \hat{D}$ by Lemma 3.3. Letting $\hat{p} : \hat{D} \to D$ be the natural projection, we note that $p_I \circ \hat{p} = p_I \circ \hat{p}_I$, and hence

$$p_I \circ \hat{p} \circ \hat{\psi} = p_I \circ \hat{p}_I \circ \psi = p_I \circ \varphi_1,$$

so that $\psi := \hat{p} \circ \hat{\psi}$ is a solution to $(p_I : D \to D/I, p_I \circ \varphi_1 : G_{F_\pi} \to D/I)$.

Since $\hat{p} : \hat{D} \to D$ maps I to itself, ψ is totally ramified. Thus, the extension E_π/F_π fixed by $\ker(\hat{p} \circ \psi)$ is a D-extension with inertia group I. In particular, its residue extension N/M at $s \mapsto s_\pi$ is the extension fixed by the kernel of $\varphi_0 := p_I \circ \varphi_1$. Since η_π has kernel $C_D(I)$ containing I, the map η_π factors as $\overline{\eta}_\pi \circ p_I$ for an epimorphism $\overline{\eta}_\pi : D/I \to V_\pi$. Thus the kernel of $\overline{\eta}_\pi \circ \varphi_0 = \eta_\pi \circ \varphi_1 = \varphi$ is $M(\mu_e)$. Hence N contains $M(\mu_e)$. Moreover, as $\ker \overline{\eta}_\pi = p_I(C_D(I)) = C_D(I)/I$, we have $N^{C_D(I)} = M(\mu_e)$, giving (c). Since φ_1 is $K(\mu_e)$-regular, so are φ_0 and N/M, giving (b). □

Remark 3.5. If D is a p-group, where either p is an odd prime or $\mu_4 \subseteq K$, we can further choose $M = K(\mu_e)^{V_\pi}(t)$. Indeed, the same proof applies when replacing the application of Lemma 3.1 by Lemma 3.2.

Corollary 3.6. *Let* $\pi = (I, D)$ *be a split ID pair over a number field* K *with* $C_D(I) = I$, *and* $\eta_\pi : D \to V_\pi$ *as above. Then there exist infinitely many places* $t \mapsto t'_\pi$ *of* $K(t)$ *such that the completion* F_π *at* $t \mapsto t'_\pi$ *admits a* D*-extension* E_π/F_π *with inertia group* I *and residue extension* $K(\mu_e)/K(\mu_e)^{V_\pi}$.

Proof. Letting t'_π be a primitive element for $M := K(\mu_e)^{V_\pi}$, and F_π the completion at $t \to t'_\pi$. Letting $\varphi : G_M \to \mathrm{Gal}(M(\mu_e)/M) \cong D/I$ be the restriction map and $p_2 : D \to D/I$ the projection modulo I, the assertion follows from Lemma 3.3 applied to the split embedding problem (φ, p_2). □

3.2. **Specialization.** The following lemma provides our induction step from parametrizing the residue (D/I)-extensions of $K_\mathfrak{p}$ to parametrizing D-extensions of $K_\mathfrak{p}$.

Let $M/K(t)$ be a finite extension, and C a finite group. We shall say that an extension N/M with constant field $L = N \cap \overline{K}$ has the property $P(C, \alpha, \beta)$ if for every prime $\mathfrak{p}$ of K which is inert in L, and every unramified C-extension $U_\mathfrak{p}/K_\mathfrak{p}$, there are at least $\alpha N(\mathfrak{p}) - \beta$ (resp. infinitely many if $N(\mathfrak{p}) = \infty$) residue classes $\bar{t}_0$ mod $\mathfrak{p}$, and for each class infinitely many degree 1 places $t \mapsto t_0 \in K_\mathfrak{p}$ with $t_0 \equiv \bar{t}_0 \pmod{\mathfrak{p}}$ such that $U_\mathfrak{p}$ is the specialization of N at $t \mapsto t_0$.

Proposition 3.7. *Let K be the fraction field of a Dedekind domain R, and t, s indeterminates. Let $F_0 := K(t)$ and $E/F_0(s)$ an extension. Assume that the completion of $E/F_0(s)$ at $s \mapsto s_\pi$ has Galois group D, inertia group I, and its residue extension N/M has $P(D/I, \alpha, \beta)$ for $\alpha, \beta > 0$, and constant field L. Then $E/F_0(s)$ specializes to all tame D-extensions of $K_\mathfrak{p}$ with inertia group I, for every prime $\mathfrak{p}$ that is inert in L/K and has sufficiently large norm $N(\mathfrak{p})$.*

Proof. We start by choosing a "nice" polynomial with splitting field E over $K(t, s)$. Starting with a separable polynomial $P \in K(t, s)[x]$ with splitting field E, by a suitable change of the variables s and t, we may assume each of the following:

i) P is monic with coefficients in $R[t, s]$.
ii) The minimal polynomial $Q(t, s)$ of s_π over $K(t)$ lies in $R[t, s]$ and is monic of positive degree in both s and t.

Indeed, rendering the polynomials P and Q monic over the respective rings $R[t, s]$ and $R[t]$ is elementary; additionally to obtain a monic polynomial Q in t (without touching the other conditions), substitute $s + t^k$ for s, for sufficiently large k.

Next, we introduce the constants relevant to the proof. Let γ be the t-degree of Q. By Lemma 2.4 and Remark 2.5, there exists a constant $d = d_P$, depending only on P, such that there are at most d residue classes of s_0 mod $\mathfrak{p}$ for which $\mathfrak{p}$ becomes a bad prime in $\mathcal{S}_R(P_{s_0})$ for $P_{s_0} = P(t, s_0, x) \in R[t, x]$.

Let $T_\mathfrak{p}/K_\mathfrak{p}$ be a D-extension with inertia group I at a prime $\mathfrak{p} \lhd R$ with norm $N(\mathfrak{p}) \geq (\beta + \gamma d)/\alpha$ such that $\mathfrak{p}$ is inert in L. We claim that $T_\mathfrak{p}/K_\mathfrak{p}$ is a specialization of $E/K(t, s)$. Set $U_\mathfrak{p} = T_\mathfrak{p}^I$. Let $\mathcal{T}_\mathfrak{p} \subseteq R/\mathfrak{p}$ denote the set of residue classes $\bar{t}_0$ of t_0 mod $\mathfrak{p}$ where $U_\mathfrak{p}/K_\mathfrak{p}$ is a specialization of N/M for infinitely many $t \mapsto t_0$ of residue $\bar{t}_0$. Note that $|\mathcal{T}_\mathfrak{p}| \geq \alpha N(\mathfrak{p}) - \beta$ by $P(D/I, \alpha, \beta)$ and $\alpha N(\mathfrak{p}) - \beta \geq 1$ by choice of $\mathfrak{p}$. For each $\bar{t}_0 \in \mathcal{T}_\mathfrak{p}$, choose $t_0 \in K_\mathfrak{p}$ with residue $\bar{t}_0$ such that:

(1) $U_\mathfrak{p}/K_\mathfrak{p}$ is the residue extension of N/M at an unramified degree 1 prime $\mathfrak{P}$ of $MK_\mathfrak{p}$ over $(t - t_0) \lhd K_\mathfrak{p}[t]$;
(2) $(t - t_0)$ is not in the finite set $\mathcal{S}_{K_\mathfrak{p}[t]}(P)$ in Theorem 2.2, where P is viewed as a polynomial in x with parameter s over the ring $K_\mathfrak{p}[t]$;
(3) t_0 is not one of the finitely many ramification points of the nonconstant algebraic function $s_\pi \in \overline{K_\mathfrak{p}(t)} \setminus \overline{K_\mathfrak{p}}$.[8]

Since $\mathfrak{P}$ is an unramified degree 1 prime over $(t - t_0) \lhd K_\mathfrak{p}[t]$, there exists a unique $s_0 \in K_\mathfrak{p}$ such that $v_\mathfrak{P}(s_\pi - s_0) > 0$. Denote by $E_{s_0}/K_\mathfrak{p}(t)$ its residue extension at

[8]Note that all these ramification points are roots of the discriminant of $Q(t, s) \in R[t][s]$.

$s \mapsto s_0$. By (3), we have $s_\pi - s_0 \not\equiv 0 \bmod \mathfrak{P}^2$, and hence $v_\mathfrak{P}(s_\pi - s_0) = 1$. Since in addition $(t - t_0) \notin \mathcal{S}_{K_\mathfrak{p}[t]}(P)$ by (2) and the residue extension of $NK_\mathfrak{p}/MK_\mathfrak{p}$ at $t \mapsto t_0$ is $U_\mathfrak{p}/K_\mathfrak{p}$ by (1), Theorem 2.2 implies that, at $t \mapsto t_0$, $E_{s_0}/K_\mathfrak{p}(t)$ has decomposition group D, inertia group I, and residue extension $U_\mathfrak{p}/K_\mathfrak{p}$.

To specialize t, we first note that as s_π is a root of Q and $Q(t, s_0)$ is not the zero polynomial mod $\mathfrak{p}$ by condition ii) above, the residue $\overline{t}_0$ is a root of $Q(t, \overline{s}_0) \in R/\mathfrak{p}[t]$, for every value $\overline{s}_0 \in R/\mathfrak{p}$. Thus, running over the values t_0 picked above, we see that every residue s_0 mod $\mathfrak{p}$ is obtained from at most γ values t_0, each with distinct residues $\overline{t}_0 = t_0 \bmod \mathfrak{p}$. Since there are $\alpha N(\mathfrak{p}) - \beta$ choices for $\overline{t}_0$, and $(\alpha N(\mathfrak{p}) - \beta)/\gamma > d$, we may choose t_0 and s_0 such that $\mathfrak{p}$ is not a bad prime in $\mathcal{S}_R(P_{s_0})$ for $P_{s_0} = P(t, s_0, x) \in R[t, x]$. Then by Theorem 2.6 and Remark 2.7, there exist $t_1 \in K_\mathfrak{p}$ (with $v_\mathfrak{p}(t_1 - t_0)$ positive and coprime to e) such that the specialization of $E_{s_0}/K_\mathfrak{p}(t)$ at $t \mapsto t_1$ is $T_\mathfrak{p}/K_\mathfrak{p}$. □

Remark 3.8. We note that the proof shows that the number of possible mod $\mathfrak{p}$ residues of specialization values (t_1, s_0) is at least $\alpha' N(\mathfrak{p}) - \beta'$ (or infinite if $N(\mathfrak{p}) = \infty$), where $\alpha', \beta' > 0$ are constants that depend only on E/K, α and β. In particular, as soon as $\alpha' N(\mathfrak{p}) - \beta' > 0$, the set of possible values $(t_1, s_0) \in \mathbb{A}^2_{K_\mathfrak{p}}$ is Zariski dense (namely, containing infinitely many "good" values t_1, and for each such value infinitely many good values s_0).

Finally for a number field K, we show that a K-regular extension has $P(C, \alpha, \beta)$ for certain cyclic subgroups C, and constants $\alpha, \beta > 0$.

Lemma 3.9. *Let K be a number field and $M/K(t)$ a finite K-regular extension. Let N/M be a Galois extension with constant field $L = N \cap \overline{K}$, and $C \leq \mathrm{Gal}(N/M)$ a cyclic subgroup which projects onto $\mathrm{Gal}(LM/M)$. Then N/M has $P(C, \alpha, \beta)$ for some constants $\alpha, \beta > 0$ depending only on N/K.*

Proof. Let $\mathfrak{p}$ be a prime of K which is inert in L. Note that $MK_\mathfrak{p}$ is $K_\mathfrak{p}$-regular, and $LK_\mathfrak{p}$ is the constant field of $NK_\mathfrak{p}$. Thus, $Gal(NK_\mathfrak{p}/MK_\mathfrak{p})$ is isomorphic to $\mathrm{Gal}(N/M)$. Let $\Omega_N/K(t)$ be the Galois closure of $N/K(t)$. Then in particular, the composition of the restriction map with the above isomorphism gives an epimorphism $p : \mathrm{Gal}(\Omega_N K_\mathfrak{p}/MK_\mathfrak{p}) \to \mathrm{Gal}(N/M)$. Letting x be a generator of C, choose $\widehat{x} \in p^{-1}(x)$. Then by Theorem 2.8 and Remark 2.9, there exist at least $\alpha N(\mathfrak{p}) - \beta$ residue classes $\overline{t}_0$ mod $\mathfrak{p}$, and infinitely many places $t \mapsto t_0 \in K_\mathfrak{p}$ of each residue, at which $\Omega_N K_\mathfrak{p}/K_\mathfrak{p}(t)$ specializes to an unramified extension of $K_\mathfrak{p}$ with group $\langle \widehat{x} \rangle$, for $\alpha, \beta > 0$ depending only on N/K. In particular, $MK_\mathfrak{p}/K_\mathfrak{p}(t)$ specializes to the trivial extension, and $NK_\mathfrak{p}/MK_\mathfrak{p}$ specializes to an extension with group $p(\langle \widehat{x} \rangle) = C$. □

The following corollary combines Proposition 3.7 and Lemma 3.9 to show that the extension constructed in Corollary 3.4 (with an enlarged base field K) specializes to all but finitely many tame local D-extensions with inertia group I.

For an ID pair (I, D) with $e := |I|$, let $\eta_\pi : D \to \mathrm{Gal}(K(\mu_e)/K)$ be the associated map in Lemma 2.1, and let V_π be its image.

Corollary 3.10. *Let $\pi = (I, D)$ be an ID pair over a number field K, and $e := |I|$. Let $E/K(t, s)$ be a Galois extension whose completion at a place $s \mapsto s_\pi$ is an extension E_π/F_π, as in Corollary 3.4*[9]*, such that its residue has constant extension $K(\mu_e)/K(\mu_e)^{V_\pi}$. Then there exists $\delta > 0$, depending only on E/K, such that every tame D-extension of $K_\mathfrak{p}$ with inertia group I is a specialization of $E/K(t, s)$, for every prime $\mathfrak{p}$ of K with $N(\mathfrak{p}) > \delta$.*

Proof. Let $\mathfrak{p} \in P_K(\pi)$, that is, such that there exists a D-extension of $K_\mathfrak{p}$ with inertia group I. By Lemma 2.1.(c), the Frobenius element of $\mathfrak{p}$ is a generator of V_π. In particular, a prime $\mathfrak{P}$ of $K_0 := K(\mu_e)^{V_\pi}$ over $\mathfrak{p}$ is of degree 1, and is inert in $K(\mu_e)$. Note that $K_0 \subseteq (K_0)_\mathfrak{P} \cong K_\mathfrak{p}$.

Let N/M be the residue extension of $EK_0/K_0(t, s)$. By construction, N/M has cyclic Galois group D/I, and its constants extension is $K(\mu_e)/K_0$. Thus, N/M has $P(D/I, \alpha, \beta)$ over K_0 for some $\alpha, \beta > 0$ depending only N/K, by Lemma 3.9. As $\mathfrak{P}$ is inert in $K(\mu_e)$, Proposition 3.7 and Remark 3.8 give constants α', β', depending only on E/K, such that every tame local D-extension with inertia group I of $K_\mathfrak{p}$ is a specialization over $K_\mathfrak{p}$ of $EK_0/K_0(t, s)$ and hence of $E/K(t, s)$, when $\alpha' N(\mathfrak{p}) - \beta' > 0$. The assertion follows for $\delta := \beta'/\alpha'$. □

In case the ID-pair (I, D) is split and $C_D(I) = I$, it is even simper to deduce that the extension in Corollary 3.6 specializes to every D-extension with inertia group I. Let η_π and V_π be as in Lemma 2.1.

Corollary 3.11. *Let $\pi = (I, D)$ be a split ID pair over a number field K with $C_D(I) = I$. Let $e := |I|$, and $\eta_\pi : D \to V_\pi$ be as above. Let $E/K(t)$ be a Galois extension whose completion at a place $t \mapsto t_\pi$ is an extension E_π/F_π with residue extension $K(\mu_e)/K(\mu_e)^{V_\pi}$. Then every tame D-extension of $K_\mathfrak{p}$ with inertia group I is a specialization of $E/K(t)$, for every prime $\mathfrak{p} \notin \mathcal{S}_0(E/K(t))$.*

Proof. As in Corollary 3.10, let $\mathfrak{p} \in P_K(\pi) \setminus \mathcal{S}_0(E/K(t))$, so that a prime $\mathfrak{P}$ of $L := K(\mu_e)^{V_\pi}$ over $\mathfrak{p}$ is of degree 1, is inert in $K(\mu_e)$, and has completion $L_\mathfrak{P} \cong K_\mathfrak{p} \supseteq L$. Let T be a tame D-extension of $K_\mathfrak{p} \cong L_\mathfrak{P}$ with inertia group I. As $I = C_D(I)$, Lemma 2.1.(2) implies that $T^I \cong L_\mathfrak{P}(\mu_e)$. As $\mathfrak{P}$ is inert in $L(\mu_e)/L$, the extension $T^I/L_\mathfrak{P}$ is the compositium of $L(\mu_e)/L$ with $L_\mathfrak{P}$. As $\mathfrak{p} \notin \mathcal{S}_0$, there exist (infinitely many) $t_0 \in K_\mathfrak{p} \cong L_\mathfrak{P}$ such that the specialization of $EL_\mathfrak{P}/L_\mathfrak{P}(t)$ at $t \mapsto t_\pi$ is $T/K_\mathfrak{p}$, by Theorem 2.6. □

[9]The construction in 3.4 is applied here with K replaced by $K(\mu_e)^{V_\pi}$

4. Proof of Theorem 1.1 and further rationality analysis

4.1. **Proof of Theorem 1.1(2a).** Let $\Pi = \Pi_{G,K}$ denote the set of all ID pairs (I, D) over K with $D \leq G$. Let $\eta_\pi : D \to V_\pi$ be the map associated to $\pi \in \Pi$ in Lemma 2.1. By Corollary 3.4, we may choose distinct places $s \mapsto s_\pi$, $\pi \in \Pi$ of $K(t)(s)$, and extensions E_π/F_π of the completion F_π of $K(t, s)$ at $s \mapsto s_\pi$ satisfying the following. For $\pi = (I, D)$, the extension E_π/F_π has Galois group D, inertia group I, and residue extension N/M satisfying:

(a) $M/K(\mu_e)^{V_\pi}(t)$ is a finite and M is $K(\mu_e)^{V_\pi}$-regular, where $e := |I|$;
(b) the constant field of N is $K(\mu_e)$;
(c) $N^{C_D(I)} \cong M(\mu_e)$, where $C_D(I)$ acts via the natural quotient $D \to D/I$.

Since G has a generic extension over K, [31, Theorem 5.8] gives a G-extension $E/K(t, s)$ with decomposition group conjugate to D and completion E_π/F_π at $s \mapsto s_\pi$, for every $\pi = (I, D) \in \Pi$. By Corollary 3.10, every tame D-extension $T_\mathfrak{p}/K_\mathfrak{p}$ with inertia group I is a specialization of $E/K(t, s)$ at some $t \mapsto t'_\pi$, $s \mapsto s'_\pi$, for every prime $\mathfrak{p}$ of K with $N(\mathfrak{p}) > \delta$, where δ is a constant depending only on E/K.

4.2. **Proof of Theorem 1.1(1).** Passing from Galois extensions of étale algebras to their underlying fields, the essential local dimension $\mathrm{ld}_K(G)$ (resp. parametric dimension $\mathrm{pd}_K(G)$) of G over K is the minimal integer $d \geq 0$ for which there exist G-extensions E_i/F_i, $i = 1, \dots, r$, such that every field H-extension of $K_\mathfrak{p}$ (resp. of K) for $H \leq G$, is a specialization of some E_i/F_i, for all but finitely many primes $\mathfrak{p}$.

Lemma 4.1. *Assume G has essential local (resp., essential parametric) dimension $\leq d$ over K. Then the same holds for any subgroup of G.*

Proof. Let E/F be a G-extension of function fields such that K is the constant field of F. For $H \leq G$, let L/K be an H-extension, and assume that ν is a place of FL at which the residue extension of EL/FL is a field H-extension M/L. The latter implies that there is a degree-1 place ω of $E^H L$ over ν such that the residue of $EL/E^H L$ at ω is M/L. The assertion now follows by choosing $L = K$ for parametric dimension or $L = K_\mathfrak{p}$ for primes $\mathfrak{p}$ of K for local dimension. □

In particular, since any finite group embeds as a subgroup into a group possessing a generic extension (such as a suitable symmetric group), Lemma 4.1 and Theorem 1.1(2a) imply Theorem 1.1(1).

The argument in Lemma 4.1 does not apply in the Hilbert-Grunwald dimension setting, since it is not clear how to construct K-rational places of E^H for $H \leq G$. However, it yields a weak version of the Hilbert–Grunwald property, see Remark 4.5.

4.3. **Proof of Theorem 1.1(2b).** Theorem 1.1(2b) is derived from (2a) using:

Lemma 4.2. *Let K be a number field, G a finite group and E/F a G-extension such that F is purely transcendental over K, and such that for all primes $\mathfrak{p}$ of K outside of some finite set T, every G-extension of $K_\mathfrak{p}$ occurs as a specialization of E/F at some place away from the branch locus of E/F. Then all Grunwald problems for G away from T have a field solution within the set of specializations of E/F.*

Proof. Suppose $F = K(\mathbf{t})$, where $\mathbf{t} = (t_1, \dots, t_r)$ are indeterminates. Let $P(\mathbf{t}, x)$ be a defining polynomial of E/F, separable in x, and let $\mathbf{t} \to \mathbf{t}_\mathfrak{p} \in K_\mathfrak{p}^r$, $\mathfrak{p} \in S \setminus T$ be places at which E/F specializes to prescribed extensions $L^{(\mathfrak{p})}/K_\mathfrak{p}$. For all cyclic subgroups $C \leq G$, we add (unramified) C-extensions $L^{(\mathfrak{p})}/K_\mathfrak{p}$ over distinct primes $\mathfrak{p} \notin S \cup T$, to the Grunwald problem, and enlarge S to contain these primes.

We claim that there must then in fact be infinitely many choices for $\mathbf{t}_p \in K_\mathfrak{p}^r$ points with the above property, for each $\mathfrak{p} \in S \setminus T$. Indeed, as in Section 2.5, letting $f : X \to Y$ be a G-cover over K with function fields E/F and $\psi_\mathfrak{p} : G_{K_\mathfrak{p}} \to G$ an epimorphism representing $L^{(\mathfrak{p})}/K_\mathfrak{p}$, the point $\mathbf{t}_\mathfrak{p}$ gives rise to a $K_\mathfrak{p}$-rational point on $\tilde{X}$, where $f^{\psi_\mathfrak{p}} : \tilde{X} \to \mathbb{P}^1$ is the twist of f by $\psi_\mathfrak{p}$. Since $K_\mathfrak{p}$ is an "ample" field, a.k.a. "large" field, every smooth absolutely irreducible variety with a simple point has infinitely many points, proving the claim. We may therefore without loss assume $P(\mathbf{t}_\mathfrak{p}, x)$ to be separable. This makes Krasner's lemma applicable, yielding full mod-$\mathfrak{p}^m$ residue classes (for some $m > 0$) of good values $\mathbf{t}_\mathfrak{p}$ in $K_\mathfrak{p}^r$. By the Chinese remainder theorem we may pick (infinitely many) $\mathbf{t}_0 \in K^r$ in the above residue classes, so that $E_{\mathbf{t}_0}/K$ has the prescribed completions at all $\mathfrak{p} \in S$. Finally, since we extended our Grunwald problem, the Galois group of $E_{\mathbf{t}_0}/K$ must intersect every conjugacy class of cyclic subgroups in G, and thus equal G by Jordan's theorem. □

In particular, if there exists a single extension E/F, with F purely transcendental of transcendence degree $\mathrm{ld}_K(G)$, that specializes to G-extensions of $K_\mathfrak{p}$ for $\mathfrak{p}$ away from a finite set, then $\mathrm{hgd}_K(G) = \mathrm{ld}_K(G)$.

Theorem 1.1(2b) now follows by simply choosing E/F as in Theorem 1.1(2a).

Remark 4.3. Note that it is possible that the equality $\mathrm{hgd}_K(G) = \mathrm{ld}_K(G)$ holds in general, however already $\mathrm{hgd}_K(G) < \infty$ implies a positive answer to the IGP for G over K. Indeed, by adding suitable local extensions to a Grunwald problem, one may force its solutions to be fields as in the proof of Lemma 4.2.

Remark 4.4. The existence of a generic extension is used in the proof of Theorem 1.1 in order to find a G-extension $E/K(t, s)$ whose completions at $s \mapsto s_\pi$ are given extensions E_π/F_π, for $\pi \in \Pi$.

Embedding G into SL_n, and letting $X := SL_n/G$, it is well known that $SL_n \to X$ is a versal G-torsor[10]. In this setting, the property required for Theorem 1.1(2b) is

[10] One may also consider the map $\mathbb{A}^n \to \mathbb{A}^n/G$, and take a smooth open subset of $\mathbb{A}^n/G$ to form a G-torsor.

the following weak approximation property. Namely, the restriction

$$X(F_0(s)) \to \prod_{\pi\in\Pi} X(F_\pi),$$

has to have dense image, where $F_0 := K(t)$, and F_π is the completion of $F_0(s)$ at a place $s \mapsto s_\pi$. This weak approximation property clearly holds if X is rational, but fails in general in view of Remark A.2.

4.4. **Some strengthenings.** We note some strengthenings of the assertions of Theorem 1.1, which directly follow from the proof above.

Remark 4.5. The proof of Lemma 4.1 and Theorem 1.1(2b) give the following for every finite group G:

There exists a constant $d = d_G$, a G-extension E/F with F of transcendence degree 2, and a finite set T of primes of K such that every Grunwald problem $L^{(\mathfrak{p})}/K_\mathfrak{p}$, $\mathfrak{p} \in S \setminus T$, is solvable over an overfield K' of degree $[K' : K] \le d$ in the following sense: For every prime $\mathfrak{p} \in S\setminus T$, there is a primes $\mathfrak{p}'$ of K' with $K'_{\mathfrak{p}'} \cong K_\mathfrak{p}$, such that the Grunwald problem $L^{(\mathfrak{p})}/K'_{\mathfrak{p}'}, \mathfrak{p} \in S \setminus T$ is solvable within the set of K'-rational specializations of E/F.

Indeed, embedding G into a group Γ possessing a generic extension, let $E/K(t,s)$ be the Γ-extension provided by Theorem 1.1(2b), and $F = E^G$. As in the proof of Lemma 4.1, the property that $E/K(t,s)$ specializes to every H-extension $L^{(\mathfrak{p})}/K_\mathfrak{p}$ for $H \le G$, implies that there are (distinct) $K_\mathfrak{p}$-rational places $\nu(\mathfrak{p})$, $\mathfrak{p} \in S \setminus T$ of $K_\mathfrak{p}(t,s)$, and places $\omega(\mathfrak{p})$ of $FK_\mathfrak{p}$ of degree 1 over $\nu(\mathfrak{p})$, such that the residue of E/F at $\omega(\mathfrak{p})$ is $L^{(\mathfrak{p})}/K_\mathfrak{p}$, for every $\mathfrak{p} \in S\setminus T$. As in the proof of Lemma 4.2, there exists a K-rational place ν of $K(t,s)$ approximating $\nu(\mathfrak{p}), \mathfrak{p} \in S\setminus T$, and a place ω of degree of F over ν such that the residue of $EK_\mathfrak{p}/FK_\mathfrak{p}$ at ω is $L^{(\mathfrak{p})}/K_\mathfrak{p}$, for every $\mathfrak{p} \in S\setminus T$. Here the degree of ω over ν is bounded merely by $d = d_G := [F : K(t,s)]$.

Remark 4.6. The proof of Theorem 1.1 shows the following stronger conclusion:

Given a number field K and a finite group G with a generic extension over K, there exists a G-extension $E/K(t,s)$ (namely, as constructed above) such that *for each finite extension* $K' \supseteq K$ and each prime $\mathfrak{p}'$ of K' outside of some finite set, every G-extension of $K'_{\mathfrak{p}'}$ occurs as a $K'_{\mathfrak{p}'}$-specialization of $E/K(t,s)$. In other words the extension $EK'/K'(t,s)$ yields the conclusion of Theorem 1.1(2a) over K', for each finite extension $K' \supseteq K$. To see this, we first claim that every ID pair for G over K' is also an ID pair for G over K, by Lemma 2.1: The composition of the map $\eta'_\pi : D \to \mathrm{Gal}(K'(\mu_e)/K')$ with the restriction $\mathrm{Gal}(K'(\mu_e)/K') \to \mathrm{Gal}(K(\mu_e)/K' \cap K(\mu_e))$ is $\eta_\pi : D \to \mathrm{Gal}(K(\mu_e)/K)$. By Chebotarev's theorem, there exist (infinitely many) primes $\mathfrak{p}$ of K of norm q such that $\langle\sigma_{q,e}\rangle = \mathrm{Im}(\eta_\pi)$. Thus, the "if" direction of Lemma 2.1.(2) implies that (I, D) is an ID pair over K, as claimed. Now the base change

$EK'/K'(t,s)$ still has a completion E_π/F_π satisfying the conditions of Corollary 3.4. Thus the proof of Theorem 1.1(2a) applies over K'. The same strengthening applies to the conclusion on solvability of Grunwald problems in Theorem 1.1(2b).

Finally note that there is a notion of "arithmetic dimension", similar to parametric dimension, which allows "finite base change" [30]. Namely in this context, it measures the number of parameters required to parametrize all étale algebra G-extensions of K', where K' runs over all finite extensions of K. The above remark suggests that the "local" version of arithmetic dimension is then also 2. Thus in similarity to Remark A.3 the study of arithmetic dimension in this context also concerns the extent to which local global principles fail.

4.5. **Allowing finitely many parametrizing extensions.** The following remark suggests the possibility of extending the definition of local dimension to cover all specializations of K_ν, where ν runs over all places of K.

Remark 4.7. Let T be the finite set from Theorem 1.1. There are only finitely many Galois extensions of $L^{(\nu)}/K_\nu$ with Galois group a subgroup of G, where ν runs through the infinite places and the places corresponding to primes of T. If each of the above extensions $L^{(\nu)}$ is the completion $L_\nu \cong L^{(\nu)}$ of some G-extension L/K at ν, we may add the extensions $L(t,s)/K(t,s)$ and the extension given by Theorem 1.1 to a finite list of extensions whose specializations contain all extension of K_ν with Galois group a subgroup of G, where ν runs over all places of K. Moreover, in view of the conjectured Beckmann–Black lifting property, it is expected that the extensions $L(t,s)/K(t,s)$ can be replaced by finitely many K-regular G-extensions.

Finding extensions L/K as above is possible in many cases, e.g. whenever G is of odd order [29, Theorem 9.5.9], but it is not always possible. For example, if $G=\mathbb{Z}/8$, then there is no G-extension of $\mathbb{Q}$ whose completion at 2 is the unramified G-extension $L^{(2)}/\mathbb{Q}_2$. Thus, there exists no G-extension $E/\mathbb{Q}(t,s)$ which specializes to $L^{(2)}/\mathbb{Q}_2$. However, very few such extensions $L^{(\mathfrak{p})}/K_\mathfrak{p}$ are known to exist.

Remark 4.8. Replacing the finite extension $E/K(t,s)$ in Theorem 1.1.(2a) by finitely many G-extensions $E(\pi)/K(t,s)$, $\pi\in\Pi$, the weak approximation property in Remark 4.4 can be replaced by the mere requirement that each of the extensions E_π/F_π is the completion at $s\mapsto s_\pi$ of some G-extension $E(\pi)/K(t,s)$. We do not know of a case in which this property fails.

Take for example G to be a p-group, and K to be a number field containing the $|G|$-th roots of unity. Remark 3.5 shows that $s\mapsto s_\pi$ is a K-rational place. Thus, for an ID-pair (I,D), the field E_π is of the form $M((s-s_\pi))(\sqrt[e]{\alpha(s-s_\pi)})$, where $e=|I|$, $M/K(t)$ is a (D/I)-extension, and $\alpha\in M$. In this case $M(s,\sqrt[e]{\alpha(s-s_\pi)})/K(t,s)$ is a D-extension which induces the desired G-extension of étale algebras $E(\pi)/K(t,s)$ whose completion at $s\mapsto s_\pi$ is E_π/F_π. In fact, we expect that the rigidity method can be used, for certain p-groups G and number fields K as above, to show that the extension $E(\pi)/K(t,s)$ can be chosen to be a K-regular field extension.

Appendix A. Parametric dimension

The following example was noticed in [30, Theorem 3.4]. It shows that $\mathrm{pd}_{\mathbb{Q}}(G)$ may be strictly smaller than $\mathrm{ed}_{\mathbb{Q}}(G)$, based on Lagrange's four square theorem.

Example A.1. Let $G = (\mathbb{Z}/2)^5$. It is well known that $\mathrm{ed}_{\mathbb{Q}}(G) = 5$. We show here that $\mathrm{pd}_{\mathbb{Q}}(G) \leq 4$. Namely, we consider the extensions[11] $E_{i,\varepsilon} := \mathbb{Q}(t_1,\dots,t_4,\sqrt{\varepsilon(t_1+\cdots+t_i)})$, $i \leq 4$, $\varepsilon \in \{1,-1\}$ of $F := \mathbb{Q}(t_1,\dots,t_4)$, and claim that every G-extension $L/\mathbb{Q}$ is a specialization of at least one of these extensions. It is well known that L is of the form $\mathbb{Q}(\sqrt{a_1},\dots,\sqrt{a_5})$ for some $a_i \in \mathbb{Q}^\times$, $i = 1,\dots,4$. (Note that $a_i = 0$ may be replaced by $a_i = 1$ without changing the extension).

We consider two cases according to whether all a_i's are positive or at least one of them is not. In the former case set $\varepsilon = +1$, and in the latter $\varepsilon = -1$. Then the quadratic form $Q_\varepsilon(x_1,\dots,x_5) := a_1x_1^2 + \cdots + a_4x_4^2 - \varepsilon a_5 x_5^2$ has a nontrivial solution $(x_1,\dots,x_5)$. By possibly reordering the a_i's, we may assume $x_1,\dots,x_i$ and x_5 are nontrivial for some $1 \leq i \leq 4$. Then the specialization of $E_{i,-\varepsilon}$, at $t_j \mapsto a_j x_j^2$ for $j \leq i$, and $t_j \mapsto a_j$ for $i < j \leq 4$, is $\mathbb{Q}(\sqrt{a_1},\dots,\sqrt{a_4},\sqrt{-\varepsilon\sum_{j=1}^i a_j x_j^2})$. The last field is L since $-\varepsilon\sum_{j=1}^i a_j x_j^2 \in \mathbb{Q}^\times$ and $a_5 \in \mathbb{Q}^\times$ are equivalent mod $(\mathbb{Q}^\times)^2$ by our choice of $x_1,\dots,x_i$ and x_5.

The following remark motivates allowing more than one extension in the definition of parametric dimension[12] over number fields:

Remark A.2. The following example is constructed in a joint work with D. Krashen, and is an (explicit) version of [7, Proposition A.3] over number fields. There exist p-groups G and number fields K, e.g. with $G = \mathbb{Z}/8$, and $K = \mathbb{Q}(\sqrt{17})$, with the following property. There exist two primes $\mathfrak{p}_1,\mathfrak{p}_2$ of K and Galois extensions $L^{(\mathfrak{p}_i)}/K_{\mathfrak{p}_i}$ with groups $H_i \leq G$ (resp. G-extensions L_i/K), $i = 1,2$ such that there is no G-extension $E/K(t)$ that specializes to both $L^{(\mathfrak{p}_i)}/K_{\mathfrak{p}_i}$ (resp. L_i/K), $i = 1,2$.

If there was a single extension $E/K(\mathbf{t})$, $\mathbf{t} = (t_1,\dots,t_r)$ which specializes at $t \mapsto \mathbf{t}_i$ to $L^{(\mathfrak{p}_i)}/K_{\mathfrak{p}_i}$, $i = 1,2$ (resp. every extension L/K) with Galois group a subgroup of G, then by finding a line $t_i = t_i(t)$, $i = 1,\dots,r$ passing through $\mathbf{t}_1$ and $\mathbf{t}_2$, we would obtain an extension $E_1/K(t)$ that specializes to $L^{(\mathfrak{p}_i)}/K_{\mathfrak{p}_i}$ (resp. L_i/K), $i = 1,2$,

[11] The number of extensions is clearly not minimal, and it is interesting to see it is possible to parametrize all G-extensions using a single extension of transcendence degree 1.

[12] The remark considers extensions of rational function fields. Since we do not require rational function fields in the definition, it remains to be seen whether there is an example of a number field K and group G such that $\mathrm{pd}_K(G)$ would increase upon demanding $r = 1$ in the definition.

obtaining a contradiction. Therefore, one cannot expect the existence of such an extension for any finite number of parameters r. On the other hand for $G = \mathbb{Z}/8$, it is known [28] that there exist finitely many polynomials over $K(t_1, \ldots, t_5)$ which parametrize all $\mathbb{Z}/8$-extensions.

Moreover, in view of [8, §10] and [32, §2.4], it is possible that there is a uniform bound $b = b_{G,K}$ satisfying the following. For every finitely many primes $\mathfrak{p}_i$, $i = 1, \ldots, r$, and Galois extensions $L^{(\mathfrak{p}_i)}/K_{\mathfrak{p}_i}$, $i = 1, \ldots, r$ (resp. L_i/K) with group a subgroup of G, there exist at most b extensions of $K(t)$ that specialize to each of the extensions $L^{(\mathfrak{p}_i)}/K_{\mathfrak{p}_i}$, $i = 1, \ldots, r$ (resp. L_i/K). In particular, this suggests that by replacing a single extension by finitely many extensions, one may be able to parametrize all Galois extensions of $K_\mathfrak{p}$ (resp. of K) with group $\leq G$.

Finally, the following remark defines local global principles such that $\mathrm{pd}_K(G) \leq 2$ whenever the principles hold. First note that we expect the following "morphisms" version of Theorem 1.1(1) also holds for every finite group G and number field K: There exists a a morphism $\varphi : G_F \to G$, where F is of transcendence degree 2 over K, such that the set of specializations $\varphi_{t_0} : G_K \to G$ of φ surjects onto $\mathrm{Hom}(G_{K_\mathfrak{p}}, G)$ under the restriction map, for every prime $\mathfrak{p}$ away from a finite set T.

Remark A.3. In the setting of Section 2.5, the constructed extension E/F in Theorem 1.1 can be replaced by a dominant Galois morphism $f : X \to Y$ of smooth irreducible 2-dimensional varieties defined over K with Galois group G. Let $\varphi : \pi_1(Y') \to G$ be an epimorphism representing f. Twisting f by $\psi : G_K \to G$, we obtain a morphism $f^\psi : \tilde{X}^\psi \to Y$, with the property that $\tilde{X}^\psi$ has a K-rational point if and only if ψ is a specialization of φ. On the other hand, letting $\psi_\mathfrak{p} : G_{K_\mathfrak{p}} \to G$ denote the restriction of ψ to $G_{K_\mathfrak{p}}$, we may also consider the twist $f^{\psi_\mathfrak{p}} : \tilde{X}^{\psi_\mathfrak{p}} \to Y$ (which is the base change of f^ψ to $K_\mathfrak{p}$).

In this setting, the above "morphisms" version of Theorem 1.1 implies that outside a finite set of primes T (depending on X, but not on ψ), the variety $\tilde{X}^\psi$ has a $K_\mathfrak{p}$-rational point for $\mathfrak{p} \notin T$. In fact if we choose ψ whose completions ψ_ν are known to appear as specializations of φ, when ν runs through places in T and infinite places, then $X^\psi(K_\nu) \neq \emptyset$ for such ν. For such ψ, if the local global principle

$$\prod_\nu \tilde{X}^\psi(K_\nu) \neq \emptyset \Rightarrow \tilde{X}^\psi(K) \neq \emptyset,$$

holds where ν runs over all places of K, then ψ is a specialization of φ. Similarly, to obtain this for all ψ, so that $\mathrm{pd}_K(G) \leq 2$, one needs the local global principle

$$\prod_{\mathfrak{p} \notin T} \tilde{X}^\psi(K_\mathfrak{p}) \neq \emptyset \Rightarrow \tilde{X}^\psi(K) \neq \emptyset,$$

where $\mathfrak{p}$ runs over primes of K not in T.

In similarity to [23], the expectation that $\mathrm{pd}_K(G) > 2$ for many groups G therefore implies a broad failure of the local global principle.

Appendix B. Groups of essential local dimension 1

The following result shows that the bound $\mathrm{ld}_K(G) \leq 2$ is in general sharp. It follows largely from [24]:

Theorem B.1. *Let G be a finite group containing at least one noncyclic abelian subgroup. Then G has local dimension at least 2 over K.*

In [24], the following was shown:

Proposition B.2. *Let K be a number field, G a finite group, and $E_i/K(t)$, $i = 1, \dots, r$ finitely many Galois extensions with group G. Then there exists a finite extension L of K such that, for all primes $\mathfrak{q}$ of K which split completely in L, for all $i \in \{1, \dots, r\}$ and for all non-branch points $t_0 \in \mathbb{P}^1(K_\mathfrak{q})$ of L, the specialization $(E_i \cdot K_\mathfrak{q})_{t_0}/K_\mathfrak{q}$ is cyclic. In particular, if G contains a minimal non-cyclic abelian subgroup $A \cong C_p \times C_p$ (for some prime p), then for all primes $\mathfrak{q}$ of K which split completely in $L(\zeta_p)$, the field $K_\mathfrak{q}$ possesses an A-extension, whereas no $K_\mathfrak{q}$-specialization of any $E_i/K(t)$ has group A.*

Remark B.3. In [24], these Galois extensions $E_i/K(t)$ were assumed K-regular. That assumption is however not necessary for the assertions given here. Indeed, if K' is the full constant field of a G-extension $E/K(t)$, then the first assertion holds (with K replaced by K') for the K'-regular extension $E/K'(t)$, and in particular if $\mathfrak{q}$ is any prime of K completely split in L, then any prime $\mathfrak{q}'$ extending $\mathfrak{q}$ in K' is split in L, whence $(E \cdot K_\mathfrak{q})_{t_0}/K_\mathfrak{q}$ must be cyclic for all non-branch points $t_0 \in K_\mathfrak{q}(= K'_{\mathfrak{q}'})$.

Proof of Theorem B.1. Proposition B.2 implies that no finite set of G-extensions $E_i/K(t)$, $i = 1, \dots, r$ can yield all the necessary local extensions via specializations. To prove Theorem B.1, we generalize this to finite sets of G-extensions E_i/F_i, $i = 1, \dots, r$ where F_i is a K-regular function field in one variable over K.

Since the set of primes splitting completely in finitely many given finite extensions of K is of positive density by Chebotarev's density theorem, it suffices to consider one extension E_i/F_i at a time. Thus, let G be a finite group possessing a subgroup $A \cong C_p \times C_p$ for some prime p, and let E/F be a G-extension, with F a finite K-regular extension of $K(t)$. Consider the Galois closure $\Omega/K(t)$ of $E/K(t)$. This has Galois group embedding into $G \wr H$ where H is the Galois group of the Galois closure of $F/K(t)$. Applying Proposition B.2 for $\Omega/K(t)$ gives a positive density set $\mathcal{S}$ of primes of K such that for each $\mathfrak{q} \in \mathcal{S}$, $K_\mathfrak{q}$ has an A-extension, whereas all specializations of $\Omega/K(t)$ at some $t_0 \in K_\mathfrak{q}$ have cyclic decomposition group. In particular, for every $K_\mathfrak{q}$-rational place of F, the corresponding specialization of $E \cdot K_\mathfrak{q}/F \cdot K_\mathfrak{q}$ has cyclic decomposition group, and a fortiori does not have decomposition group A. □

Remark B.4. There remains the question which finite groups precisely have local dimension 1 (over a given number field). Theorem B.1 reduces the candidate groups to the class of groups in which all Sylow subgroups are cyclic or generalized quaternion (see, e.g., [5, Chapter XI, Theorem 11.6]). In [21], we will reduce the list much further and in fact achieve a full classification of groups G of local dimension 1 over (e.g.) real number fields. Over these fields, G is either (a) cyclic of order 2 or odd prime power order; or (b) a semidirect product $C \rtimes D$ of two cyclic groups C, D as in (a), with faithful semidirect product action.

Remark B.5. It would also be interesting to investigate invariants such as ld or hgd over fields K other than number fields. Just to give one simple, yet interesting example: If $K = \mathbb{C}(x)$ is the rational function field over $\mathbb{C}$, then any finite group G has local dimension 1 over K, as a direct consequence of Riemann's existence theorem: Indeed, every local extension is now of the form $\mathbb{C}(((x-c)^{1/e}))/\mathbb{C}((x-c))$ for some $c \in \mathbb{P}^1(\mathbb{C})$.[13] Choose any G-extension $L/\mathbb{C}(t)$ in which a representative of every non-trivial conjugacy class of G occurs as an inertia group generator at some branch point. Then setting $E := L(x)$, the G-extension $E/K(t)$ reaches all local G-extensions at all (finite) primes of K, simply by specializing $t \mapsto x - c$ with suitable $c \in \mathbb{C}$. On the other hand, [12] shows that (with very few exceptions) $E/K(t)$ is very far away from reaching all G-extensions of K itself by specialization, suggesting again a discrepancy between local and parametric dimension in this case.

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