MONODROMY GROUPS OF PRODUCT TYPE

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Abstract. We complete the classification of monodromy and ramification of indecomposable covering $f : X \to \mathbb{P}^1$ of sufficiently large degree in comparison to the genus of X. This is acheived by dealing with groups of product type, the last family of groups in the Aschbacher–O'Nan–Scott structure theorem for which the problem was open.

1. Introduction

Background. The monodromy group $\mathrm{Mon}(f)$ and ramification of a branched covering $f : X \to \mathbb{P}^1$ of the complex projective line are fundamental invariants which lie at the heart of many problems in arithmetic geometry, complex analysis, and other areas, cf. [26]. Here, X is a complex curve and $\mathrm{Mon}(f)$ is the Galois group of the Galois closure of $\mathbb{C}(X)$ over the function field $\mathbb{C}(t) = \mathbb{C}(\mathbb{P}^1)$, viewed as a permutation group on $\mathrm{Hom}_{\mathbb{C}(t)}(\mathbb{C}(X), \overline{\mathbb{C}(t)})$. The ramification type of f over a point P is the multiset of ramification indices of points in $f^{-1}(P)$ over P, and the ramification type of f is the multiset of ramification types of f over all branch points.

The problem of determining the possible monodromy groups $\mathrm{Mon}(f)$ and ramification for *indecomposable* coverings $f : \mathbb{P}^1 \to \mathbb{P}^1$ originates in the work of Chisini [5], Ritt [34], and Zariski's thesis, cf. [35, 36]. These determine the possible solvable monodromy groups for indecomposable $f : \mathbb{P}^1 \to \mathbb{P}^1$ of sufficiently large degree, when the two point stabilizer is trivial. Following the classification of finite simple groups, there has been a major progress towards a solution of this problem. This progress led to a solution in the case of simple groups by the combination of Liebeck–Saxl [19], Liebeck–Shalev [20], and Frohardt–Magaard [12], and cumulated to the Guralnick–Shareshian conjecture [16, Conjecture 1.0.4] of all possible monodromy groups of indecomposable coverings $f : X \to \mathbb{P}^1$ of sufficiently large degree $\deg f \gg g_X$ in comparison to the *genus* g_X. Namely, the conjecture asserts that for $g > 0$, there exists a constant N_g such that:

For every covering $f : X \to \mathbb{P}^1$ of degree $n \geq N_g$ and genus $g_X = g$, the monodromy group $\mathrm{Mon}(f)$ is one of the following:

We thank Thao Do and Robert Guralnick for helpful discussions. The first author is grateful for the support of the Israel Science Foundation grant No. 577/15, and the NSF for support under a Mathematical Sciences Postdoctoral Fellowship. The second author thanks the NSF for support grant DMS-1162181. We also thank the United States-Israel Binational Science Foundation (BSF) for its support under Grant No. 2014173.

(1) $\mathrm{Mon}(f) = A_n$ or S_n;
(2) $\mathrm{Mon}(f) = A_d$ or S_d with $n = d(d-1)/2$;
(3) $A_d^2 \le \mathrm{Mon}(f) \le (S_d^2) \rtimes C_2$, where $n = d^2$, and the semidirect action of S_2 permutes the two copies of S_d;
(4) $\mathrm{Mon}(f)$ is a subgroup of a semidirect product $(C_p)^2 \rtimes C_k$, for $k \in \{2, 3, 4, 6\}$, where C_p is the cyclic group of order p, and p is a prime, see [26, Table 9.1] for a complete list.

This paper and its companion [26] complete the proof of the Guralnick–Shareshian conjecture, and determine all possible ramification types for f in each of the cases (2)-(4), see [26, Theorem 1.1]. Case (1) is the generic case which occurs for all $f : X \to \mathbb{P}^1$ as above whose ramification does not appear in cases (2)-(4). We note that in case (4) the ramification types were already known to Zariski, cf. [17, Proposition 3.8], [26, Proposition 9.5].

Main result. The proof of the above result follows the Aschbacher–O'Nan–Scott structure theorem for primitive groups G, see [17]. The case (A) where G has an abelian minimal normal subgroup is treated by Guralnick–Thompson [17] and Neubauer [27]. The case (B) where G has more than one minimal normal subgroup is treated by Shih [30]. Henceforth assume G has a unique minimal normal subgroup Q, and that Q is nonabelian, so that $Q = L^t$ for some nonabelian simple group L. The case (C1) where Q acts regularely is treated by Guralnick–Thompson [17]. The case (C2) where L acts regularely is treated by Aschbacher [1]. The remaining case (C3), also known as groups of *product type*, is when G acts on a set Δ^t, inducing a pointwise action of L^t on Δ^t, with respect to a nonregular action of L on Δ.

The case $t = 1$ is completed in [26, Theorem 1.2]. In this paper, we fix any $t \ge 2$ and determine all product type monodromy groups of sufficiently large degree ℓ^t. Let $S_\ell \wr S_t = S_\ell^t \rtimes S_t$ denote the standard wreath product where the semidirect product action permutes the t-copies of S_ℓ.

Theorem 1.1. *Fix an integer $t \ge 2$. There exist positive constants $c = c_t, d = d_t$ depending only on t, such that for every indecomposable covering $f : X \to X_0$ with monodromy group G of product type acting on Δ^t, either $g_X > c\ell^{t-1} - d\ell^{t-2}$ or the ramification type of f appears in Table 3.1. Moreover, for all of the ramification types in Table 3.1, one has $t = 2$, and $A_\ell^2 \le G \le S_\ell \wr S_2$, and $g_{X_0} = 0$, and $g_X \le 1$.*

Conversely, for every type in Table 3.1, there exists an indecomposable rational function $f : X \to \mathbb{P}^1$ with monodromy group $A_\ell^2 \le G \le S_\ell \wr S_2$ with this ramification type, see Section 11.

We note that in an opposite scenario for groups of product type, when ℓ is fixed and $t > 8$, the Guralnick–Shareshian conjecture follows from Guralnick–Thompson and Guralnick–Neubauer [14, Corollary 8.7]. In combination with the latter, Theorem 1.1 completes the determination of monodromy and ramification for groups of product type.

Also note that the theorem and [14, Theorems 7.1 and 7.2] show that for indecomposable coverings $f : X \to \mathbb{P}^1$ of sufficiently large degree n, with monodromy group of product type with $t \ge 2$, whose ramification does not appear in Table 3.1, one has $g_X > c'\sqrt{n}$ for some absolute constant $c' > 0$, or equivalently $n < (g_X/c')^2$. Thus for groups of product type, the constant N_g can be chosen to be quadratic in g.

Method of proof. Let $f : X \to \mathbb{P}^1$ be a covering with monodromy group G of product type, and assume for simplicity $A_\ell^t \le G \le S_\ell \wr S_t$ with the standard action on $\{1, \dots, \ell\}^t$, for $t \ge 2$. In particular, it also acts on $\{1, \dots, t\}$ with kernel $K := G \cap S_\ell^t$. Letting H be a point stabilizer one has the following diagram of natural projections:

$$\begin{array}{ccc} Z = \tilde{X}/(H \cap K) & \xrightarrow{\pi} & \tilde{X}/H \cong X \\ {\scriptstyle h}\downarrow & & \downarrow \\ Y = \tilde{X}/K & \xrightarrow[\pi_0]{} & \tilde{X}/G \cong \mathbb{P}^1, \end{array}$$

where $Y := \tilde{X}/K$ and $Z := \tilde{X}/(H \cap K)$. The approach to proving that g_X is large is to show that g_Z is large in comparison to the difference $D := g_Z - \deg \pi \cdot g_X$, where the latter quantity can be expressed in terms of the ramification contribution of π to the Riemann–Hurwitz formula. This approach was used by Guralnick–Thompson to show that $g_Z - D \ge a_1 \ell^t$ for some absolute constant $a_1 > 0$ in all cases where $g_Y > 1$, and by Guralnick–Neubauer [14, Corollary 8.7][1] to show that $g_Z - D > a_2 \ell^{t-1}$ for some absolute constant $a_2 > 0$, in all cases where $t > 8$ and $g_Y \le 1$.

Estimating the difference $g_Z - D$ using the methods of [17, 14] becomes notoriously difficult when treating cases with $t \le 8$. Namely, thes hardest cases, where D is large, appear when one of the branch cycles of f acts on $\{1, \dots, t\}$ as a transposition, i.e. only when $t = 2, 3, 4, 6$, and 8, cf. Section 6. In fact, in some of these cases there are (infinite families of) ramification data $\mathcal{R}$, that is multisets of partitions of ℓ^t, such that if there was a covering $f : X \to \mathbb{P}^1$ with monodromy group $A_\ell^t \le G \le S_\ell \wr S_t$ and ramification $\mathcal{R}$ then the Riemann–Hurwitz formula would imply that $g_Z - D < a_3$ for some absolute constant $a_3 > 0$. However, it turns out that none of these ramification data correspond to a covering, by Section 2.4, as there is no corresponding product 1 tuple.

In contrast to [17, 14, 26], for small values of t our approach starts with exploiting the product 1 relation associated to a covering, cf. Section 2.4. When $g_Y \le 1$, this product one relation in $S_\ell \wr S_t$ amounts to a single product one relation in S_ℓ, which by Riemann's existence theorem gives rise to a new degree ℓ covering $\hat{f} : \hat{Y} \to \mathbb{P}^1$ with transitive monodromy group. The existence of the new covering $\hat{f}$ gives severe restrictions on the ramification of f via the Riemann–Hurwitz formula for $\hat{f}$ (or alternatively Ree's inequality). This is carried out in Section 8.

A key ingredient in the proof is showing that a bound of the form $g_X < \alpha \ell^{t-1}$, for a constant $\alpha > 0$, forces the ramification of h to be similar to the ramification of a Galois covering. More precisely, since $A_\ell^t \le K \le S_\ell^t$, h is the fiber product of degree ℓ coverings h_i, $i = 1, \dots, t$, each with monodromy A_ℓ or S_ℓ. For sufficiently large ℓ, the above bound implies that either the number of preimages in $h_i^{-1}(P)$ is bounded in terms of α, or all

[1] Although the proof of [14, Corollary 8.7] in some cases is somewhat intricate, a very simple proof can be extracted for $t > 60$, which suffices for our purposes.

preimages of P have the same ramification index with a bounded amount of exceptions, for each point P of Y and $i = 1, \ldots, t$, cf. Section 4. This part relies on an idea from [9].

Using the Riemann–Hurwitz formula for $\hat{f}$ and the ramification of almost Galois type for the coverings h_i, $i = 1, \ldots, t$, we then show that the ramification of f itself is similar to that of a Galois extension, cf. Section 8. We then analyze the possibility for such ramification types of f, in order to obtain a bound $D \leq b_4\ell^{t-2}$ and hence the desired inequality $g_Z - D > a_4\ell^{t-1} - b_4\ell^{t-2}$, for some positive constants $a_4 = a_{4,t}$, $b_4 = b_{4,t}$, depending only on t. Note that the proof of Theorem 1.1 is self contained and does not rely on [14] and [17].

2. Preliminaries and notation

2.1. **Monodromy.** Let $\mathbb{K}$ be an algebraically closed field of characteristic 0. A *covering* $f : X \to X_0$ is a morphism of (smooth projective geometrically irreducible) curves defined over $\mathbb{K}$. Letting $\tilde{f} : \tilde{X} \to X_0$ be its Galois closure, we call $\mathrm{Mon}(f) := \mathrm{Gal}(\tilde{f})$ the *monodromy group* of f. We note that $G := \mathrm{Mon}(f)$ is a permutation group on $n := \deg f$ letters via its right action on $\Omega := H\backslash G$, where H is the monodromy group of the natural projection $\tilde{X} \to X$. The group H is then a point stabilizer in this action. Given a subgroup H_2 of $\mathrm{Mon}(f)$, there is a natural projection $f_2 : \tilde{X}/H_2 \to X_0$. This natural projection is *equivalent* to f if there exists an isomorphism $\eta : X \to \tilde{X}/H_2$ such that $f = f_2 \circ \eta$. The coverings f_2 and f are then equivalent if and only if H_2 is conjugate to H. The covering f is indecomposable if and only if H is a maximal subgroup of G or equivalently if the G-action on Ω is primitive.

We note that unless otherwise mentioned all group actions on sets are right actions. Thus, permutation multiplication is left to right, e.g. $(1,2)(1,3) = (1,2,3)$. Denote $x^y = y^{-1}xy$ for any $x, y \in G$.

2.2. **Ramification.** Let P be a point of X_0. For a point $Q \in f^{-1}(P)$, let $I = I(\tilde{Q}/P)$ denote the inertia group of Q over P. Since $\mathbb{K}$ is algebraically closed, I coincides with the decomposition group $\{\sigma \in G \,|\, \tilde{Q}^{\sigma} = \tilde{Q}\}$. We call $x \in G$ a *branch cycle* of f over a point P of X_0 if x generates $I(Q/P)$ for some $Q \in f^{-1}(P)$. We write $\mathrm{Orb}_\Omega(I)$ (resp. $\mathrm{Orb}_\Omega(x)$) for the set of orbits of I (resp. x) on $\Omega = H\backslash G$.

Write $e_f(Q) := |I|$ for the *ramification index* of Q under f. For a point P of X_0 write $E_f(P)$ for the tuple $[e_f(Q_1), \ldots, e_f(Q_k)]$ where $Q_1, \ldots, Q_k$ are the preimages of P under f, ordered so that $e_f(Q_1) \geq \ldots \geq e_f(Q_k)$. We write $E_f(P) = [u_1^{k_1}, \ldots, u_r^{k_r}]$ if the entry u_i appears k_i times, $i = 1, \ldots, r$. We say Q is a *ramification point* of f if $e_f(Q) \neq 1$, and P is a *branch point* of f if $f^{-1}(P)$ contains a ramification point. We call the multiset $\{E_f(P) \,|\, P \text{ branch point of } f\}$ the *ramification type* of f.

The following basic lemma describes ramification using monodromy and inertia groups. In particular in the context of the end of Section 1, it will be used to derive the ramification and branch cycles of h from those of f.

Lemma 2.1. *Let $\tilde{f} : \tilde{X} \to X_0$ be a Galois covering with monodromy group G, and $I = I(\tilde{Q}/P)$ the inertia group of $\tilde{Q} \in \tilde{f}^{-1}(P)$, for $P \in X_0$. Let $H \leq G$ be a subgroup, $X :=$*

$\tilde{X}/H$, and let $f : X \to X_0$ and $\hat{f} : \tilde{X} \to X$ be the natural projections. Then there is a bijection between the double cosets $I\backslash G/H$ and points in $f^{-1}(P)$, given by sending $I\sigma H$ to $Q_\sigma := \hat{f}(\tilde{Q}^\sigma)$. Moreover, the ramification index $e_\sigma := e_f(Q_\sigma)$ equals $|I\sigma H|/|H|$, and $\sigma I^{e_\sigma}\sigma^{-1}$ is an inertia group of $\hat{f}$ over Q_σ.

Proof. As the action of G on $\tilde{f}^{-1}(P)$ is transitive and I is a point stabilizer in this action, this action is equivalent to the G-action on $I\backslash G$. The image of two points $\tilde{Q}^\sigma, \tilde{Q}^\tau \in \tilde{f}^{-1}(P)$ in $X = \tilde{X}/H$ coincides if and only if $\tilde{Q}^{\sigma H} = \tilde{Q}^{\tau H}$ or equivalently if $I\sigma H = I\tau H$, for $\sigma, \tau \in G$. Hence, the map $I\sigma H \to Q_\sigma = \hat{f}(\tilde{Q}^\sigma)$ is well defined and gives an inclusion $\mathrm{Orb}_\Omega(I) \to f^{-1}(P)$. It is surjective since the action of G on $\tilde{f}^{-1}(P)$ is transitive. Since the inertia group $I_{\tilde{f}}(\tilde{Q}^\sigma/P)$ is the subgroup $\sigma^{-1}I\sigma$, we have $I_{\hat{f}}(\tilde{Q}^\sigma/Q_\sigma) = \sigma^{-1}I\sigma \cap H$. Thus,

$$e_f(Q_\sigma) = [\sigma^{-1}I\sigma : \sigma^{-1}I\sigma \cap H] = \frac{|I|}{|\sigma^{-1}I\sigma \cap H|} = \frac{|I\sigma H|}{|H|},$$

and $I_{\hat{f}}(\tilde{Q}^\sigma/Q_\sigma) = \sigma^{-1}I\sigma \cap H = \sigma^{-1}I^{e_\sigma}\sigma$. □

In particular, $E_f(P)$ is the tuple of cardinalities of orbits in $\mathrm{Orb}_\Omega(I)$ in decreasing order. Note that if f is Galois then $e_f(Q)$ is independent of the choice of $Q \in f^{-1}(P)$, in which case we denote $e_f(P) := e_f(Q)$.

2.3. **Riemann–Hurwitz.** The Riemann-Hurwitz formula expresses the *genus* g_X as

$$2(g_X - 1) = 2n(g_{X_0} - 1) + \sum_{Q \in X(\mathbb{K})} R_f(P), \text{ where}$$

$$R_f(P) := \sum_{Q \in f^{-1}(P)} (e_f(Q) - 1) = \sum_{r \in E_f(P)} (r - 1) = n - |E_f(P)| = n - |\mathrm{Orb}_\Omega(x)|.$$

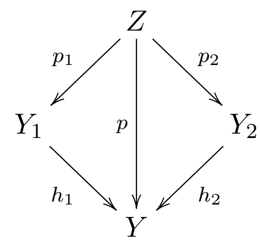
For a subset S of points of X_0, denote by $R_f(S)$ the total contribution $\sum_{P \in S} R_f(P)$ to the Riemann-Hurwitz formula.

We note that given a degree m covering $\pi : Z \to X$ and points $Q \in f^{-1}(P), R \in \pi^{-1}(Q)$, ramification indices are multiplicative $e_{f\circ\pi}(R) = e_\pi(R)e_f(Q)$, and satisfy the equality $\sum_{Q \in f^{-1}(P)} e_\pi(Q) = m$. In particular, this gives the *chain rule*:

$$\begin{aligned} R_{f\circ\pi}(P) &= \sum_{Q \in f^{-1}(P)} \sum_{R \in \pi^{-1}(Q)} \big(e_\pi(R)(e_f(Q) - 1) + e_\pi(R) - 1\big) \\ &= \sum_{Q \in f^{-1}(P)} \big(m(e_f(Q) - 1) + R_\pi(Q)\big) = mR_f(P) + R_\pi(f^{-1}(P)). \end{aligned} \tag{2.1}$$

We restate the following version of Abhyankar's lemma [26, Lemma 9.2]. We say that a covering $h : Z \to Y$ is *minimal that factors through coverings* $h_i : Y_i \to Y$, $i \in I$ if the only compositions $h = u \circ v$ for which u factors through h_i are those where v is an isomorphism, that is $u = h_i \circ \hat{h}_i$ for some coverings $\hat{h}_i$, $i \in I$ if and only if $\deg v = 1$. The greatest common divisor of two positive integers a, b is denoted by (a, b).

Lemma 2.2. *Let $h_i : Y_i \to Y$, $i = 1, 2$ be coverings, and $p : Z \to Y$ a minimal covering which factors through h_1 and h_2. Write $p = h_i \circ p_i$, $i = 1, 2$. Let P be a point of Y and $Q_i \in h_i^{-1}(P)$ a preimage, for $i = 1, 2$.*



(1) *Then $e_p(Q) = \mathrm{lcm}(e_{h_1}(Q_1), e_{h_2}(Q_2))$ for every $Q \in p_1^{-1}(Q_1) \cap p_2^{-1}(Q_2)$;*
(2) *If furthermore $\deg p = \deg h_1 \cdot \deg h_2$, then there are $(e_{h_1}(Q_1), e_{h_2}(Q_2))$ points Q in Z with image $p_i(Q) = Q_i$ for both $i = 1, 2$.*

In particular,

$$R_{p_2}(h_2^{-1}(P)) = \sum_{r_1 \in E_{h_1}(P), r_2 \in E_{h_2}(P)} (r_1, r_2)\Big(\frac{\mathrm{lcm}(r_1, r_2)}{r_2} - 1\Big)$$
$$= \sum_{r_1 \in E_{h_1}(P), r_2 \in E_{h_2}(P)} \big(r_1 - (r_1, r_2)\big).$$

We note that if the fiber product of h_1 and h_2 is irreducible, then under the assumption of (2), Z is isomorphic to the normalization of this fiber product.

Remark 2.3. Letting $\tilde{h}_1 : \tilde{Y}_1 \to Y$ be the Galois closure of a covering $h_1 : Y_1 \to Y$, we note that Lemma 2.2 implies that $e_{\tilde{h}_1}(P) = \mathrm{lcm}(E_{h_1}(P))$, for every point P of Y. Indeed, letting $G_1 := \mathrm{Mon}(h_1)$ and K_1 be a point stabilizer, $\tilde{h}_1$ is a minimal covering which factors through the natural projections $h_1^\sigma : \tilde{Y}_1/K^\sigma \to Y$, $\sigma \in G_1$, and hence the assertion follows by applying Lemma 2.2 iteratively.

2.4. **Riemann's existence theorem.** Fix a transitive subgroup $G \leq S_n$, conjugacy classes $C_1, \ldots, C_r$ of cyclic subgroups of G, and points $P_1, \ldots, P_r$ in $\mathbb{P}^1$. We call a tuple $x_1, \ldots, x_r \in G$ a *product one tuple for* G if $x_1 \cdots x_r = 1$ and $\langle x_1, \ldots, x_r \rangle = G$. By Riemann's existence theorem, there exists a Galois covering $\tilde{f} : \tilde{X} \to \mathbb{P}^1$ with monodromy group G such that C_i is the conjugacy class of an inertia group of $\tilde{f}$ over P_i for $i = 1, \ldots, r$ if and only if there exists a product 1 tuple $x_i \in C_i$, $i = 1, \ldots, r$ for G, see [33, Theorems 2.13 and 5.14].

Given such $\tilde{f}$, let $f : \tilde{X}/H \to \mathbb{P}^1$ be the natural projection where $H := G \cap S_{n-1}$ is a point stabilizer. Then $E_f(P_i)$ is the multiset of lengths of orbits of $x_i \in C_i$, $i = 1, \ldots, r$. We call the multiset of lengths of orbits of x_i, the *cycle structure* of x_i, and the multiset of all cycle structures of x_i, $i = 1, \ldots, r$ the *ramification type* of the tuple $x_1, \ldots, x_r$. Thus, $E_f(P_i)$ coincides with the cycle structure of x_i, $i = 1, \ldots, r$, and the ramification type of f coincides with the ramification type of $x_1, \ldots, x_r$.

A degree n *ramification data* is a multiset $\mathcal{R}$ of partitions $A_1, \dots, A_r$ of n consisting of positive integers. By Riemann's existence theorem, the existence of a degree n covering $f: X \to \mathbb{P}^1$ whose ramification type is $\mathcal{R}$ is equivalent to the existence of a product 1 tuple $x_1, \dots, x_r$ for G such that x_i has cycle structure A_i, for $i = 1, \dots, r$. The following basic lemma rules out ramification data from occurring as the ramification type of an indecomposable covering:

Lemma 2.4. [26, Lemma 9.1.(a,c)] *Let p be a rational prime, $f : \mathbb{P}^1 \to \mathbb{P}^1$ an indecomposable covering with monodromy group G, and P_1, P_2, P_3 be points of $\mathbb{P}^1$.*

(a) *If all entries of $E_f(P_1)$ and $E_f(P_2)$ are divisible by p, then f is Galois of degree p, and $G \cong C_p$.*
(c) *If all entries of $E_h(P_1)$ are even and there is a total of exactly two coprime to* 3 *entries in $E_h(P_2)$ and $E_h(P_3)$, then G is a quotient of A_4.*

2.5. **Wreath products and reduced forms.** We use the following standard wreath product notation, e.g. see [8, Section 2.7]. Let Δ and I be sets of cardinalities $\ell \geq 5$ and $t \geq 2$, respectively. Denote by $S_\Delta \wr S_I := S_\Delta^I \rtimes S_I$ the *wreath product* equipped with an action on the set $\Omega := \Delta^I$ of all tuples $\delta = (\delta(i))_{i \in I}$ given as follows. The semidirect product action is given by $(\sigma^{-1} a \sigma)(i) = a^\sigma(i) = a(i^{\sigma^{-1}})$ for all $a \in S_\Delta^I$, $\sigma \in S_I$ and $i \in I$. The action on Ω is given by $\delta^{a\sigma}(i) = \delta(i^{\sigma^{-1}})^{a(i^{\sigma^{-1}})}$, so that a acts pointwise $\delta^a(i) = \delta(i)^{a(i)}$, and σ acts by permuting the values of δ by $\delta^\sigma(i) = \delta(i^{\sigma^{-1}})$, for all $\delta \in \Delta^I, \sigma \in S_I, a \in S_\Delta^I$, and $i \in I$. Note that $S_\Delta \wr S_I$ is also equipped with an action on I via the projection to S_I.

We shall write $S_\ell \wr S_t$ for the group $S_\Delta \wr S_I$ when $\Delta = \{1, \dots, \ell\}$ and $I = \{1, \dots, t\}$.

To simplify orbit counts, we replace elements in S_Δ^I by conjugates of the following form.

Definition 2.5. Let $y = a\sigma$, $a \in S_\Delta^I, \sigma \in S_I$, and let O be the set of orbits of σ on I. We say y is a *reduced form* of $x \in G$ with *representatives* $\iota_\theta \in \theta$, $\theta \in O$, if $a(i) = 1$ for every $i \in I \setminus \{\iota_\theta \,|\, \theta \in O\}$, and $y = x^\tau$ for some $\tau \in S_\Delta^I$.

We note that every element is conjugate to an element in reduced form:

Lemma 2.6. *Let $x = a\sigma \in S_\Delta \wr S_\ell$, where $a \in S_\Delta^I$ and $\sigma \in S_I$. Let O be the set of orbits of σ and $\iota_\theta \in \theta$, $\theta \in O$, be representatives. Then there is an element $z \in S_\Delta^I$ such that $x^z = b\sigma$ with $b(i) = 1$ for $i \in \theta \setminus \{\iota_\theta\}$, and $b(\iota_\theta) = \prod_{k=0}^{|\theta|-1} a(\iota_\theta^{\sigma^k})$ for $\theta \in O$.*

Proof. Define $z \in S_\Delta^I$ on each orbit $\theta \in O$ iteratively. Fix $\theta \in O$ and for simplicity enumerate the orbit $\iota_\theta, \iota_\theta^{\sigma^{-1}}, \dots, \iota_\theta^{\sigma^{-(|\theta|-1)}}$ by $1, \dots, |\theta|$. Now set $z(1) := 1$ and iteratively set $z(j) := a(j)z(j-1)$ for $j = 2, \dots, |\theta|$. Then $x^z = b\sigma$ where $b := z^{-1} a z^{\sigma^{-1}} \in S_\Delta^I$, and

$$b(j) = z^{-1}(j)a(j)z^{\sigma^{-1}}(j) = z^{-1}(j)a(j)z(j^\sigma) = z^{-1}(j)a(j)z(j-1) = 1 \text{ for } j = 2, \dots, |\theta|$$

$$b(1) = a(1)z(1^\sigma) = a(1)a(1^\sigma)z(1^{\sigma^2}) = \dots = \prod_{k=0}^{|\theta|-1} a(1^{\sigma^k}), \text{ as desired.} \qquad \square$$

Remark 2.7. The proof also gives $z(1) = 1$, and moreover, it applies when replacing $S_\Delta \wr S_I$ by any wreath product $H \wr G$ of two permutation groups H and G.

2.6. **Groups of product type.** In this paper we consider primitive monodromy groups of the following type. This class of groups is denoted by (C3) in the Aschbacher-O'Nan-Scott theorem, cf., [2]. By [13, Theorem 11.2], the definition in [2] is equivalent to the following. Let $N_G(L)$ and $C_G(L)$ denote the normalizer and centralizer, respectively, of a subgroup L of G.

Definition 2.8. A faithful primitive group G with point stabilizer H is called of *product type* if it has a unique minimal normal subgroup Q, this subgroup Q decomposes as $L_1 \times \cdots \times L_t$ for isomorphic nonabelian simple groups $L_1, \ldots, L_t$ for $t \geq 1$, and the group $H \cap Q$ decomposes as $(H \cap L_1) \times \cdots \times (H \cap L_t)$ with $H \cap Q \neq 1$. Moreover for G of product type, the groups $H \cap L_i$, $i = 1, \ldots, t$ are conjugate in H and $N_H(L_i)C_G(L_i)$ is a maximal subgroup of $N_G(L_i)$, for $i = 1, \ldots, t$, [13, Theorem 11.2.(iii).(1)].

Throughout the paper, we shall identify groups of product type with subgroups of wreath products as follows. Put $I = \{1, \ldots, t\}$. For $i \in I$, let $S_\Delta^{(i)} \leq S_\Delta^I$ denote the subgroup consisting of tuples $a \in S_\Delta^I$ such that $a(j) = 1$ for all $j \in I \setminus \{i\}$.

Lemma 2.9. *A faithful primitive group G of product type is isomorphic to a subgroup $\hat{G} \leq S_\Delta \wr S_I$ whose minimal normal subgroup $\hat{Q}$ decomposes as a direct sum $\hat{Q} = \bigoplus_{i\in I}(\hat{Q} \cap S_\Delta^{(i)})$ of isomorphic nonabelian simple groups acting transitively on Δ. Moreover, the image of G in S_I is transitive, and the action of $\hat{G} \cap (S_\Delta^I \rtimes S_{I\setminus\{i\}})$ on the i-th copy of Δ is primitive.*

Remark 2.10. The proof of the lemma relies on the following well known facts:

(1) Let G be a faithful primitive group acting on Ω and N a nontrivial normal subgroup of G, then N acts transitively on Ω [8, Theorem 1.6A.v].
(2) Let L be a nonabelian simple group acting transitively on a set Δ and M the normalizer of L in S_Δ. Let I be a finite nonempty set, so that $M \wr S_I$ and L^I act on Δ^I. Let N be the normalizer of L^I in S_{Δ^I}. If N is primitive then $N = M \wr S_I$ [8, Lemma 4.5A].
(3) Suppose W acts transitively on Ω, and permutes the set of fibers of a surjection $\phi : \Omega \to \Upsilon$, and hence also acts on Υ. Suppose that the kernel $C \leq W$ of the action on Υ acts transitively on each fiber $\phi^{-1}(x)$, $x \in \Upsilon$. If H is the stabilizer of $\omega \in \Omega$, then the stabilizer of $\phi(\omega)$ is $H \cdot C$. Indeed, $\phi(\omega)^n = \phi(\omega)$ if and only if $\omega^n = \omega_2$ with $\phi(\omega_2) = \phi(\omega)$. As C is transitive on $\phi^{-1}(\phi(\omega))$, this condition is equivalent to the existence of $c \in C$ such that $\omega^{nc} = \omega$, for every $n \in N$.

Proof of Lemma 2.9. Since G is primitive, H is maximal in G and Q is transitive by Remark 2.10, so that $G = HQ$. Hence the bijection $H\backslash G \cong (Q \cap H)\backslash Q$ gives a faithful and primitive action of G on $(Q \cap H)\backslash Q$ with point stabilizer H. Here, since $(Hx)^h = Hxh = Hh^{-1}xh$, we have $((Q \cap H)x)^h = (Q \cap H)h^{-1}xh$ and $((Q \cap H) \cdot x)^q = (Q \cap H)xq$, for all $x, q \in Q$, and $h \in H$. Thus, the bijection $(H \cap Q)\backslash Q \cong (L_1 \cap H)\backslash L_1 \times \cdots \times (L_t \cap H)\backslash L_t$ gives a faithful and primitve G-action on $(L_1 \cap H)\backslash L_1 \times \cdots \times (L_t \cap H)\backslash L_t$, where H is a point stabilizer, and $Q = L_1 \times \cdots \times L_t$ acts by right multiplication.

Let $\Delta := (L_1 \cap H)\backslash L_1$, and write $I = \{1, \ldots, t\}$. Note that since G acts transitively on the copies $L_1, \ldots, L_t$ of L, and since $G = HQ$, so does H acts transitively on the copies.

Since an element of H conjugating L_1 and L_i also conjugates $L_1 \cap H$ and $L_i \cap H$, it defines an isomorphism $\phi_i : L_1 \to L_i$ and a ϕ_i-equivarient isomorphism $\Delta \cong (H \cap L_i)\backslash L_i$, for $i \in I$. Identifying each of the L_i-sets $(H \cap L_i)\backslash L_i$ with Δ via these isomorphisms for $i \in I$, we obtain a faithful and primitive action of G on $\Omega := \Delta^I$, where H is a point stabilizer and $Q = L_1 \times \cdots \times L_t$ acts on $\Omega = \Delta^t$ coordinatewise. Let $\hat{G}, \hat{H}, \hat{Q}$, and $\hat{L}_i = \hat{Q} \cap S_\Delta^{(i)}$ be the images of G, H, Q and L_i in S_Ω, respectively, for $i \in I$. Since $\hat{G}$ is contained in the normalizer N of $\hat{Q}$ in S_Ω, the subgroup N acts primitively on Ω. Since N is primitive, and $\hat{Q} = \prod_{i=1}^t \hat{L}_i$, Remark 2.10.(2) implies that $N = M \wr S_I$, where M is the normalizer of $\hat{Q} \cap S_\Delta^{(i)}$ in $S_\Delta^{(i)}$, for $i \in I$. This proves the first assertion.

Since G acts transitively on $\{L_1, \ldots, L_t\}$ by conjugation, it also acts transitively on $\{\hat{L}_i = \hat{Q} \cap S_\Delta^{(i)} : i \in I\}$ by conjugation. It is straightforward to check that this action is equivalent to the G-action on $I = \{1, \ldots, t\}$, and hence the latter is also transitive.

Fix $i \in I$. Since the normalizer of $\hat{L}_i$ in N is $M^I \rtimes S_{I\backslash\{i\}}$, we have $N_{\hat{G}}(\hat{L}_i) = \hat{G} \cap (S_\Delta^I \rtimes S_{I\backslash\{i\}})$, for $i \in I$. In particular, $N_{\hat{G}}(\hat{L}_i)$ acts on the i-th copy $\Delta^{(i)}$ of Δ in Ω. Since $\hat{L}_i$ is a nonabelian simple group, the kernel of this action is $C_{\hat{G}}(\hat{L}_i) = \hat{G} \cap (M^{I\backslash\{i\}} \rtimes S_{I\backslash\{i\}})$. Since G is of product type, this kernel acts transitively on the fibers of the projection $\Omega \to \Delta^{(i)}$, and hence Remark 2.10.(3) implies that the point stabilizer in the action of $N_{\hat{H}}(\hat{L}_i)$ on $\Delta^{(i)}$ is $N_{\hat{H}}(\hat{L}_i)C_{\hat{G}}(\hat{L}_i) = (N_{\hat{G}}(\hat{L}_i) \cap \hat{H})C_{\hat{G}}(\hat{L}_i)$. Since $N_H(L_i)C_G(L_i)$ is maximal in $N_G(L_i)$ by Definition 2.8, it follows that the point stabilizer $N_{\hat{H}}(\hat{L}_i)C_{\hat{G}}(\hat{L}_i)$ is maximal in $N_{\hat{G}}(\hat{L}_i)$, and hence $N_{\hat{G}}(\hat{L}_i) = \hat{G} \cap (S_\Delta^I \rtimes S_{I\backslash\{i\}})$ acts primitively. □

From now on, we shall identify groups of product type with subgroups $G \leq S_\Delta \wr S_I$ of a wreath product via Lemma 2.9.

Remark 2.11. We note that the proof of Theorem 1.1 for $t \geq 3$ does not use the primitivity of $G \leq S_\Delta \wr S_I$ but rather the other properties in Lemma 8.2. More specifically, the only used properties are the transitivity of $K = G \cap S_\Delta^I$ on Δ^I (which follows from that of Q), and the transitivity of G on I. The last property in Proposition 8.2, namely the primitivity of $G \cap (S_\Delta^I \rtimes S_{I\backslash\{i\}})$ on the i-th copy of Δ is used only in the proof for $t = 2$, namely in the final steps where Lemmas 2.4 and 2.13 are applied.

We next describe basic properties of primitive groups of product type:

Remark 2.12. Let $G \leq S_\Delta \wr S_I$ be a group of product type, let $K := G \cap S_\Delta^I$ be the kernel of the map $G \to S_I$, let K_i be a point stabilizer in the action of K on the i-th coordinate in Δ^I for $i \in I$, and $Q \subseteq K$ the minimal normal subgroup of G.

(1) Since $Q \cap S_\Delta^{(i)}$ acts transitively on Δ, so does $K \cap S_\Delta^{(i)}$. Hence, $[K : K_i] = \ell$, and $[K : H \cap K] = \ell^t$, for $i \in I$ and $t = |I|$. The latter also implies $[H : H \cap K] = [G : K]$.

(2) Letting $\sigma \in G$ be an element such that $i^\sigma = j$, the subgroup $\sigma^{-1}K_i\sigma$ is a point stabilizer in the action of K on the j-th coordinate of Δ^I, for $i, j \in I$. As G is transitive on I, this implies that the subgroups K_i, $i \in I$ are conjugate in G.

(3) If $|I| = 2$, then Lemma 2.9 implies that the action of $\hat{G} \cap (S_\Delta^I \rtimes S_{I\backslash\{i\}}) = K$ on each of the two copies of Δ is primitive, for $i \in I$.

We use the following lemma to ensure that a group is of product type:

Lemma 2.13. *Let $G \le S_\Delta \wr S_I$, $|\Delta| \ge 5$ be a transitive subgroup, $K := S_\Delta^I \cap G$ and assume that the projection of K to each coordinate contains A_Δ. Then*

(1) *G is primitive if and only if $K \supseteq A_\Delta^I$ and the G-action on I is transitive;*
(2) *if $|I| = 2$, either $K \supseteq A_\Delta^I$ or $K \subseteq \{(a, v^{-1}av) \,|\, a \in S_\Delta\}$ for some $v \in S_\Delta$.*

Proof. Write $\Delta = \{1, \dots, \ell\}$, $I = \{1, \dots, t\}$, let $\pi_i : K \to S_\ell$, $i \in I$ be the natural projections, and let $Q := A_\ell^t$.

First assume $K \supseteq A_\ell^t$ and G is transitive on I. We claim that there are no nontrivial intermediate subgroups $H \le M \le G$, where H is a point stabilizer. Since G acts transitively on I it acts transitively on the t copies of A_ℓ. Since in addition $G = HQ$, the action of H and hence of M on the t copies is also transitive. It follows that $\pi_i(Q \cap M)$, $i \in I$ are all isomorphic. Since $Q \cap H \cong A_{\ell-1}^t$, we have $\pi_i(Q \cap M) = A_{\ell-1}$ or A_ℓ for all $i \in I$. Thus $Q \cap M$ is a subdirect product of either t copies of A_ℓ or of t copies of $A_{\ell-1}$. Since in addition $Q \cap M$ contains $Q \cap H \cong A_{\ell-1}^t$, it follows that $Q \cap M = Q$ or $Q \cap H$. Since Q is transitive, we also have $[Q \cap M : Q \cap H] = [M : H]$ and hence either $[M : H] = 1$ or $[M : H] = [Q : Q \cap H] = [G : H]$ yielding the claim.

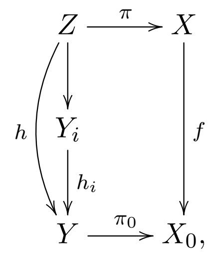
Conversely, if the G-action on I has an orbit θ which is properly contained in I, then the G-action on Δ^I is imprimitive since the blocks $U_{\delta_0} = \{\delta \in \Delta^I : \delta_{|\theta} = \delta_0\}$, $\delta_0 \in \Delta^\theta$ form a nontrivial partition. Assuming G is primitive, we let $R \le K$ be a minimal normal subgroup of G and claim that $R = Q$. As $R \lhd K$, one has $\pi_i(R) \lhd \pi_i(K)$ for all $i \in I$. Since G acts transitively on I and $\pi_i(K) \supseteq A_\ell$, the normality $\pi_i(R) \lhd \pi_i(K)$ implies that $\pi_i(R) = A_\ell$ for every $i \in I$. Thus R is a subdirect product of $A_\ell^t = Q$ and hence is isomorphic via the natural projection to A_ℓ^U for some $U \subseteq I$. Since $R \lhd G$ and G is primitive, R acts transitively on Δ^I by Remark 2.10, and hence $U = I$, proving that $R = Q$, completing the proof of (1).

Now assume $t = 2$. If $\ker \pi_1 \ne 1$, then $\pi_2(\ker \pi_1)$ is a nontrivial normal subgroup of $\pi_2(K)$. Since $\pi_2(K) \supseteq A_\ell$, this implies $\pi_2(\ker \pi_1) \supseteq A_\ell$. Thus $K \supseteq \ker \pi_1 \supseteq 1 \times A_\ell$. Since in addition $\pi_1(K) \supseteq A_\ell$, we deduce that $K \supseteq A_\ell^2$. Henceforth assume both projections π_1, π_2 are injective. Then $\pi_2 \circ \pi_1^{-1} : \pi_1(K) \to \pi_2(K)$ is an isomorphism. Identifying $\pi_1(K)$ and $\pi_2(K)$ with subgroups of S_ℓ, we get that $\pi_2 \circ \pi_1^{-1}$ is an automorphism of $A_\ell \le \pi_1(K) \le S_\ell$, and hence is given by conjugation by some $v \in S_\ell$, completing the proof of (2). □

Remark 2.14. The "only if" assertion in part (1) can be extended to cases where A_ℓ is replaced by a primitive nonabelian simple group L as follows. The same argument implies that G is primitive when one assumes that (1) G has minimal normal subgroup $Q = L_1 \times \cdots \times L_t$, where each L_i is isomorphic to the primitive nonabelian simple group L; (2) $1 \ne H \cap Q = (H \cap L_1) \times \cdots \times (H \cap L_t)$; and (3) G acts transitively on the copies $L_1, \dots, L_t$.

2.7. **Setup.** The following setup is used throughout the proof of Theorem 1.1.

Let $f : X \to X_0$ be an indecomposable covering with monodromy group $G \le S_\Delta \wr S_I$ of product type and let $\tilde{f} : \tilde{X} \to X_0$ be its Galois closure. As in Section 2.6, we let $\Omega := \Delta^I$, let $H \le G$ be a point stabilizer in the G-action on Ω, let $K := G \cap S_\Delta^I$, and let K_i be a point stabilizer in the action of K on the i-th copy of Δ in Ω, for $i \in I$. As sketched in Section 1, we let $Z := \tilde{X}/(H \cap K)$, $Y := \tilde{X}/K$, and $Y_i := \tilde{X}/K_i$ for $i \in I$. We then have the following diagram of natural projections:



for $i \in I$. Throughout the paper we let $t := |I|$ be a fixed integer which is at least 2. Also, put $m := \deg \pi_0$, and $\ell := |\Delta|$.

Remark 2.15. (1) By Remark 2.12.(1), $\deg f = \ell^t$, $\deg h = [K : H \cap K] = \ell^t$, $\deg h_i = [K : K_i] = \ell$ for $i \in I$, and $\deg \pi = m$. In particular the fiber product of h_i and h_j is irreducible for every pair of distinct $i, j \in \{1, \dots, t\}$.

(2) Fix $1 \in I$. By Remark 2.12.(2), there exists $\sigma_i \in G$ such that $K_1^{\sigma_i} = K_i$, $i \in I$. Such σ_i induces an automorphism $\overline{\sigma}_i : Y \to Y$ sending the orbit of K on $\tilde{P} \in \tilde{X}$ to the orbit of $K = \sigma_i^{-1} K \sigma_i$ on $\sigma_i^{-1}(\tilde{P})$. Similarly, σ_i induces an isomorphism $\sigma_i^* : Y_1 \to Y_i$, which sends the orbit of K_1 on $\tilde{P} \in \tilde{X}$ to the orbit of $K_i = \sigma_i^{-1} K_1 \sigma_i$ on $\sigma_i^{-1}(\tilde{P})$, so that $h_i \circ \sigma_i^* = \overline{\sigma}_i \circ h_1$. Hence, $E_{h_1}(P) = E_{h_i}(\overline{\sigma}_i(P))$ for $P \in Y$.

(3) For $t = 2$, the action of K on each copy of Δ in Ω is primitive by Remark 2.12.(3), and hence h_i is indecomposable for $i \in I$.

For $t = 2$, we shall specify the ramification type of f by writing the conjugacy class in $S_\ell \wr S_2$ of a branch cycle over each branch point of f. For $s \in S_2$, and $\alpha_i \in S_\ell$ with cycle structure A_i, $i = 1, 2$, we write the conjugacy class of $(\alpha_1, \alpha_2)s$ in $S_\ell \wr S_2$ as $(A_1, A_2)s$.

Remark 2.16. Assume $x \in S_\ell \wr S_2$ is conjugate in $S_\ell \wr S_2$ to a branch cycle of f at $P \in X_0(\mathbb{K})$. If $x = (u, 1)s$ then Lemma 2.1 implies that $x^2 = (u, u)$ is conjugate in $S_\ell \wr S_2$ and hence also in S_ℓ^2 to a branch cycle of h. Moreover, every element in $S_\ell \wr S_2 \setminus S_\ell^2$ is conjugate to an element of the form $(u, 1)s$ by Lemma 2.6. If on the other hand $x = (a, b) \in S_\ell^2$, then Lemma 2.1 implies that (a, b) and (b, a) are conjugate in S_ℓ^2 to branch cycles of h over Q_1, Q_2, respectiviely, where $\pi_0^{-1}(P) = \{Q_1, Q_2\}$.

We deduce that for $t = 2$, the ramification of f is determined by the ramification of h_1, h_2 and π_0. That is, letting $Q \in \pi_0^{-1}(P)$, the ramification type of f at P is $(E_{h_1}(Q), 1)s$ if P is a branch point of π_0, and it is $(E_{h_1}(Q), E_{h_2}(Q))$ otherwise.

3. The ramification types in Theorem 1.1

In Table 3.1 below we list the ramification types in Theorem 1.1 corresponding to indecomposable coverings $f : X \to \mathbb{P}^1$ with monodromy group $A_\ell^2 \le G \le S_\ell \wr S_2$ of product type. More specifically, we list the conjugacy classes in $S_\ell \wr S_2$ of branch cycles $x_1, \dots, x_r$

over the branch points $P_1, \dots, P_r$ of f. As in Setup 2.7, we represent elements (resp. conjugacy classes) in $G \le S_\ell \wr S_2$ by $(a_1, a_2)s^j$ (resp. $(u_1, u_2)s^j$) where $a_1, a_2 \in S_\ell$ (resp. u_1, u_2 are partitions of ℓ), s is the swap in S_2, and $j \in \{0, 1\}$.

We note that if $f : X \to \mathbb{P}^1$ is an indecomposable covering with monodromy group $G \le S_\ell \wr S_2$ such that the ramification of f appears in Table 3.1, then G is of product type with a minimal normal subgroup A_ℓ^2. For this, as in Section 2.4 it suffices to show:

Lemma 3.1. *Let $G \le S_\ell \wr S_2$, $\ell \ge 9$, be a group of product type, and $x_1, \dots, x_r$ a product 1 tuple generating G whose ramification type appears in Table 3.1. Then $G \supseteq A_\ell^2$.*

Remark 3.2. The proof relies on Jordan's theorem [8, Theorem 3.3.E] and its consequence [8, Example 3.3.1]. These assert that a primitive subgroup of S_ℓ containing either a p-cycle for a prime $p < \ell - 2$, or a product of two 2-cycles if $\ell \ge 9$, contains all of A_ℓ.

Proof of Lemma 3.1. For each ramification type in Table 3.1, by considering the powers x_j^a, $a \le 6$, for some $j \in \{1, \dots, r\}$, we see that $K := G \cap S_\ell^2$ contains an element (a_1, a_2) such that one of its coordinates a_i is a 3-cyclc, or a 2-cycle, or a product of two disjoint 2-cycles. Since G is of product type, the projection of K to both coordinates is primitive by Remark 2.12.(3). As the image of one of the projections is primitive and contains a_i, it contains A_ℓ by Remark 3.2. Conjugating the preimage of A_ℓ by an element in $G \setminus K$, we get that the images of projections on both coordinates contain A_ℓ. As G is primitive, Lemma 2.13 implies that $G \supseteq A_\ell^2$. □

The existence of coverings with such ramification types is proved in Section 11.

Remark 3.3. If a covering $f : X \to \mathbb{P}^1$ has one of the ramification types I1A.1-I1A.3 then $g_X = 1$, and for any other ramification type in Table 3.1 one has $g_X = 0$. The genus is computed in each case using Lemma 9.1. Here, the ramification of h_1 and h_2 is found by Remark 2.16 and g_{Y_1} is computed using the Riemann–Hurwitz formula for h_1.

4. Almost Galois ramification

A key ingredient in the proof of Theorem 1.1 is proving that an "almost Galois/regular" behavior occurs. For a covering $h_1 : Y_1 \to Y$, this means that similarly to Galois coverings the ramification type $E_{h_1}(P)$ has equal entries with a bounded amount of exceptions. We use the following proposition to guarantee such a property. This proposition is based on an idea from Do–Zieve [9].

Proposition 4.1. *For every integer $\alpha > 0$, there exists a constant $L_{1,\alpha}$ depending only on α that satisfies the following property. Let $h_1 : Y_1 \to Y$ and $h_2 : Y_2 \to Y$ be two coverings of degree $\ell \ge L_{1,\alpha}$, and let $p : Z \to Y$ be a minimal covering that factors through h_1 and h_2. Assume $g_Z < \alpha\ell$ and $\deg p = \deg h_1 \cdot \deg h_2$. Then for every point P of Y one of the following holds:*

(*a*) *there exists some $k \le 6$ such that, under each of the maps h_1 and h_2, the number of preimages of P which have ramification index k is at least*
$$\ell/k - 2(\alpha+1)(k+1) - 2(\alpha+1)(k^2-1)/3;$$

TABLE 3.1. Ramification types of coverings $f : X \to \mathbb{P}^1$ with monodromy group $A_\ell^2 \le G \le S_\ell \wr S_2$ of product type. We follow the notation of Setup 2.7. Here $\ell \ge 7$, s is the swap in S_2, and a is any integer satisfying $0 < a < \ell/2$ and $(a, \ell) = 1$. The notation u^v means that u appears v times in the multiset.

$I1.1$	$([\ell],[1^\ell])s, ([a,\ell-a],[1^\ell])s, ([2,1^{\ell-2}],[1^\ell])$
$I1A.1$	$([\ell],[\ell]), ([3,1^{\ell-3}],[1^\ell]), s, s$
$I1A.2a$	$([\ell],[\ell]), ([2,1^{\ell-2}],[1^\ell])s, ([2,1^{\ell-2}],[1^\ell])s$
$I1A.2b$	$([\ell],[\ell]), ([2,1^{\ell-2}],[1^\ell]), ([2,1^{\ell-2}],[1^\ell])s, s$
$I1A.2c$	$([\ell],[\ell]), ([2,1^{\ell-2}],[1^\ell]), ([2,1^{\ell-2}],[1^\ell]), s, s$
$I1A.3$	$([\ell],[\ell]), ([2^2,1^{\ell-4}],[1^\ell]), s, s$
$I1A.4$	$([a,\ell-a],[a,\ell-a]), ([3,1^{\ell-3}],[1^\ell]), s, s$
$I1A.5a$	$([a,\ell-a],[a,\ell-a]), ([2,1^{\ell-2}],[1^\ell])s, ([2,1^{\ell-2}],[1^\ell])s$
$I1A.5b$	$([a,\ell-a],[a,\ell-a]), ([2,1^{\ell-2}],[1^\ell]), ([2,1^{\ell-2}],[1^\ell])s, s$
$I1A.5c$	$([a,\ell-a],[a,\ell-a]), ([2,1^{\ell-2}],[1^\ell]), ([2,1^{\ell-2}],[1^\ell]), s, s$
$I1A.6$	$([a,\ell-a],[a,\ell-a]), ([2^2,1^{\ell-4}],[1^\ell]), s, s$
$I1A.7a$	$([\ell],[a,\ell-a]), ([2,1^{\ell-2}],[1^\ell]), s, s$
$I1A.7b$	$([\ell],[a,\ell-a]), ([2,1^{\ell-2}],[1^\ell])s, s$
$I2.1a$	$([\ell],[1^\ell])s, s, ([1^3,2^{(\ell-3)/2}],[1,2^{(\ell-1)/2}]), ([1^{\ell-2},2],[1^\ell])$
$I2.1b$	$([\ell],[1^\ell])s, ([1^{\ell-2},2],[1^\ell])s, ([1^3,2^{(\ell-3)/2}],[1,2^{(\ell-1)/2}])$
$I2.2a$	$([\ell],[1^\ell])s, ([1^{\ell-2},2],[1^\ell])s, ([1^2,2^{\ell/2-1}],[1^2,2^{\ell/2-1}])$
$I2.2b$	$([\ell],[1^\ell])s, s, ([1^2,2^{\ell/2-1}],[1^2,2^{\ell/2-1}]), ([1^{\ell-2},2],[1^\ell])$
$I2.3$	$([\ell],[1^\ell])s, s, ([2^{(\ell-3)/2},3],[1^3,2^{(\ell-3)/2}])$
$I2.4$	$([\ell],[1^\ell])s, s, ([1,2^{\ell/2-2},3],[1^2,2^{\ell/2-1}])$
$I2.5$	$([\ell],[1^\ell])s, s, ([1^2,2^{(\ell-5)/2},3],[1,2^{(\ell-1)/2}])$
$I2.6$	$([\ell],[1^\ell])s, s, ([1,2^{(\ell-5)/2},4],[1^3,2^{(\ell-3)/2}])$
$I2.7$	$([\ell],[1^\ell])s, s, ([1^2,2^{(\ell-6)/2)},4],[1^2,2^{(\ell-2)/2}])$
$I2.8$	$([\ell],[1^\ell])s, s, ([1^3,2^{(\ell-7)/2},4],[1,2^{(\ell-1)/2}])$
$I2.9a$	$([a,\ell-a],[1^\ell])s, ([1^{\ell-2},2],[1^\ell])s, ([1^2,2^{(\ell-2)/2}],[2^{\ell/2}])$
$I2.9b$	$([a,\ell-a],[1^\ell])s, s, ([1^2,2^{(\ell-2)/2}],[2^{\ell/2}]), ([1^{\ell-2},2],[1^\ell])$
$I2.10a$	$([a,\ell-a],[1^\ell])s, s, ([1,2^{(\ell-1)/2)}],[1,2^{(\ell-1)/2}]), ([1^{\ell-2},2],[1^\ell])$
$I2.10b$	$([a,\ell-a],[1^\ell])s, ([1^{\ell-2},2],[1^\ell])s, ([1,2^{(\ell-1)/2}],[1,2^{(\ell-1)/2}])$
$I2.11$	$([a,\ell-a],[1^\ell])s, s, ([1^2,2^{(\ell-6)/2},4],[2^{\ell/2}])$
$I2.12$	$([a,\ell-a],[1^\ell])s, s, ([1,2^{(\ell-5)/2},4],[1,2^{(\ell-1)/2}])$
$I2.13$	$([a,\ell-a],[1^\ell])s, s, ([2^{(\ell-4)/2},4],[1^2,2^{(\ell-2)/2}])$
$I2.14$	$([a,\ell-a],[1^\ell])s, s, ([2^{(\ell-3)/2},3],[1,2^{(\ell-1)/2}])$
$I2.15$	$([a,\ell-a],[1^\ell])s, s, ([1,2^{(\ell-4)/2},3],[2^{\ell/2}])$
$F2.1$	$([1,2,3^{(\ell-3)/3}],[3^{\ell/3}]), ([1,2,3^{(\ell-3)/3}],[1^\ell])s, s$
$F2.2$	$([2,3^{(\ell-2)/3}],[1^2,3^{(\ell-2)/3}]), ([2,3^{(\ell-2)/3}],[1^\ell])s, s$
$F2.3$	$([1,3^{(\ell-1)/3}],[2^2,3^{(\ell-4)/3}]), ([1,3^{(\ell-1)/3}],[1^\ell])s, s$
$F3.1$	$([1,3,4^{(\ell-4)/4}],[4^{\ell/4}]), ([1^2,2^{(\ell-2)/2}],[1^\ell])s, s$
$F3.2$	$([2,3,4^{(\ell-5)/4}],[1,4^{(\ell-1)/4}]), ([1,2^{(\ell-1)/2}],[1^\ell])s, s$
$F3.3$	$([3,4^{(\ell-3)/4}],[1,2,4^{(\ell-3)/4}]), ([1,2^{(\ell-1)/2}],[1^\ell])s, s$

TABLE 3.1. Continued.

$F1A.1a$	$([1^2, 2^{(\ell-2)/2}], [2^{\ell/2}]), ([1^2, 2^{(\ell-2)/2}], [1^2, 2^{(\ell-2)/2}]), ([2, 1^{\ell-2}], [1^\ell])s, s$
$F1A.1b$	$([1^2, 2^{(\ell-2)/2}], [2^{\ell/2}]), ([1^2, 2^{(\ell-2)/2}], [1^2, 2^{(\ell-2)/2}]), ([2, 1^{\ell-2}], [1^\ell]), s, s$
$F1A.2a$	$([1, 2^{(\ell-1)/2}], [1^3, 2^{(\ell-3)/2}]), ([1, 2^{(\ell-1)/2}], [1, 2^{(\ell-1)/2}]), ([2, 1^{\ell-2}], [1^\ell])s, s$
$F1A.2b$	$([1, 2^{(\ell-1)/2}], [1^3, 2^{(\ell-3)/2}]), ([1, 2^{(\ell-1)/2}], [1, 2^{(\ell-1)/2}]), ([2, 1^{\ell-2}], [1^\ell]), s, s$
$F1A.3a$	$([3, 2^{(\ell-3)/2}], [1^3, 2^{(\ell-3)/2}]), ([1, 2^{(\ell-1)/2}], [1, 2^{(\ell-1)/2}]), s, s$
$F1A.3b$	$([3, 2^{(\ell-3)/2}], [1, 2^{(\ell-1)/2}]), ([1^3, 2^{(\ell-3)/2}], [1, 2^{(\ell-1)/2}]), s, s$
$F1A.4a$	$([1, 3, 2^{(n-4)/2}], [2^{\ell/2}]), ([1^2, 2^{(\ell-2)/2}], [1^2, 2^{(\ell-2)/2}]), s, s$
$F1A.4b$	$([1, 3, 2^{(\ell-4)/2}], [1^2, 2^{(\ell-2)/2}]), ([2^{\ell/2}], [1^2, 2^{(\ell-2)/2}]), s, s$
$F1A.5$	$([1, 2^{(\ell-1)/2}], [1^2, 3, 2^{(\ell-5)/2}]), ([1, 2^{(\ell-1)/2}], [1, 2^{(\ell-1)/2}]), s, s$
$F1A.6a$	$([1, 4, 2^{(\ell-5)/2}], [1^3, 2^{(\ell-3)/2}]), ([1, 2^{(\ell-1)/2}], [1, 2^{(\ell-1)/2}]), s, s$
$F1A.6b$	$([1, 4, 2^{(\ell-5)/2}], [1, 2^{(\ell-1)/2}]), ([1^3, 2^{(\ell-3)/2}], [1, 2^{(\ell-1)/2}]), s, s$
$F1A.7a$	$([4, 1^2, 2^{(\ell-6)/2}], [2^{\ell/2}]), ([1^2, 2^{(\ell-2)/2}], [1^2, 2^{(\ell-2)/2}]), s, s$
$F1A.7b$	$([4, 1^2, 2^{(\ell-6)/2}], [1^2, 2^{(\ell-2)/2}]), ([2^{\ell/2}], [1^2, 2^{(\ell-2)/2}]), s, s$
$F1A.8$	$([1, 2^{(\ell-1)/2}], [1^3, 4, 2^{(\ell-7)/2}]), ([1, 2^{(\ell-1)/2}], [1, 2^{(\ell-1)/2}]), s, s$
$F1A.9$	$([1^2, 2^{(\ell-2)/2}], [4, 2^{(\ell-4)/2}]), ([1^2, 2^{(\ell-2)/2}], [1^2, 2^{(\ell-2)/2}]), s, s$
$F4.1$	$s, s, ([1^{n-2}, 2], 1)s, ([1^{n-2}, 2], 1)s$
$F4.2$	$s, s, s, ([1^{n-2}, 2], 1)s, ([1^{n-2}, 2], 1)$
$F4.3$	$s, s, s, s, ([1^{\ell-2}, 2], 1), ([1^{n-2}, 2], 1)$
$F4.4$	$s, s, s, s, ([1^{\ell-2}, 2], [1^{\ell-2}, 2])$
$F4.5$	$s, s, s, s, ([1^{\ell-4}, 2^2], 1)$
$F4.6$	$s, s, s, s, ([1^{\ell-3}, 3], 1)$

(b) *for each $k \le 6$, under each of the maps h_1 and h_2 the number of preimages of P which have ramification index k is at most $2(\alpha+1)(k+1)$.*

Proof. Write $E_{h_1}(P) = [e_1, \dots, e_n]$ and $E_{h_2}(P) = [f_1, \dots, f_m]$. By the Riemann–Hurwitz formula for the natural projection $p_2 : Z \to Y_2$, and Abhyankar's lemma 2.2:

$$2\ell(\alpha+1) - 2 \ge R_{p_2}(h_2^{-1}(p)) = \sum_{i=1}^{\ell}\sum_{j=1}^{m}\big(e_i - (e_i, f_j)\big). \tag{4.1}$$

We prove the assertion by induction on k, with the base case $k = 0$ being vacuous. So fix a positive integer $k \le 6$, and assume that for any $0 < k_0 < k$ the number of copies of k_0 amongst the e_i's and f_j's is at most $2(\alpha+1)(k_0+1)$. Let r be the number of i's for which e_i equals k, and let $R = \sum_{i:e_i<k} e_i$, and likewise let s be the number of j's for which f_j equals k, and $S = \sum_{j:f_j<k} f_j$. Then

$$R \le \sum_{k_0=1}^{k-1} 2(\alpha+1)(k_0+1)k_0 = 2(\alpha+1)\Big(\frac{(k-1)k(2k-1)}{6} + \frac{(k-1)k}{2}\Big) = 2(\alpha+1)\frac{k^3-k}{3}.$$

Putting $d := 2(\alpha+1)(k^3-k)/3$, and considering the contribution to the right side of (4.1) coming from the s distinct j's for which f_j equals k, we get
(4.2)
$$2(\alpha+1)\ell > s\sum_{i=1}^{n}(e_i-(k,e_i)) \geq s\sum_{e_i>k}(e_i-k) \geq s\sum_{e_i>k}\frac{e_i}{k+1} = s\frac{\ell-kr-R}{k+1} \geq s\frac{\ell-kr-d}{k+1}.$$
Likewise we get $S \leq d$ and
$$2(\alpha+1)\ell > r\frac{\ell-ks-S}{k+1} \geq r\frac{\ell-ks-d}{k+1}. \tag{4.3}$$
First assume $s \leq (\ell-d)/(2k)$. Then (4.3) gives
$$2(\alpha+1)\ell > r\frac{\ell-d}{2(k+1)}, \text{ so that } r < \frac{4(\alpha+1)\ell(k+1)}{\ell-d}$$
and thus $r < 8(\alpha+1)(k+1)$, for every $\ell > 280(\alpha+1) \geq 4(\alpha+1)(k^3-k)/3 = 2d$. Substituting this into (4.2) gives
$$2(\alpha+1)\ell > s\frac{\ell-kr-d}{k+1} > s\frac{\ell-8(\alpha+1)k(k+1)-d}{k+1},$$
so that
$$s < \frac{2(\alpha+1)\ell(k+1)}{\ell-8(\alpha+1)k(k+1)-d}, \tag{4.4}$$
for $\ell > 476(\alpha+1) \geq 8(\alpha+1)k(k+1)+d$. Since s is an integer, there exists a constant $L'_{1,\alpha}$ such that for every $\ell \geq L'_{1,\alpha}$, (4.4) forces $s \leq 2(\alpha+1)(k+1)$. Since we have shown that $r < 8(\alpha+1)(k+1)$, it follows that $r \leq (\ell-d)/(2k)$, for $\ell > 812(\alpha+1) \geq 16(\alpha+1)k(k+1)+d$ Interchanging the roles of r and s yields $r \leq 2(\alpha+1)(k+1)$.

Next assume that $s > (\ell-d)/(2k)$. Then (4.2) gives
$$2(\alpha+1)\ell > s\frac{\ell-kr-R}{k+1} \geq \frac{\ell-d}{2k}\cdot\frac{\ell-kr-R}{k+1},$$
so that
$$\ell-kr-R < \frac{4(\alpha+1)\ell k(k+1)}{\ell-d},$$
for $\ell > 140(\alpha+1) \geq d$. Hence $\ell-kr-R < 8(\alpha+1)k(k+1)$ for $\ell > 280(\alpha+1) \geq 2d$, so that
$$r > \frac{\ell-R-8(\alpha+1)k(k+1)}{k} \geq \frac{\ell-d-8(\alpha+1)k(k+1)}{k}.$$
Substituting this into (4.3) gives
$$2(\alpha+1)\ell > r\frac{\ell-ks-S}{k+1} \geq \frac{\ell-d-8(\alpha+1)k(k+1)}{k}\cdot\frac{\ell-ks-S}{k+1},$$
so that
$$\ell-ks-S < \frac{2(\alpha+1)\ell k(k+1)}{\ell-d-8(\alpha+1)k(k+1)}, \tag{4.5}$$

for $\ell > 476(\alpha+1) \geq 8(\alpha+1)k(k+1)+d$. As the left hand side of (4.5) is an integer,

$$\ell - ks - S \leq 2(\alpha+1)k(k+1) \text{ for } \ell \geq L'_{1,\alpha},$$

so that

$$s \geq \frac{\ell - S - 2(\alpha+1)k(k+1)}{k} \geq \frac{\ell - d - 2(\alpha+1)k(k+1)}{k}.$$

Since we showed that $r > (\ell - d - 8(\alpha+1)k(k+1))/k$, in particular $r > (\ell-d)/(2k)$, for $\ell > 812(\alpha+1) \geq 16(\alpha+1)k(k+1)+d$. Interchanging the roles of r and s, we obtain $r \geq (\ell - d - 2(\alpha+1)k(k+1))/k$ for every $\ell \geq L_{1,\alpha} := \max\{L'_{1,\alpha}, 812(\alpha+1)\}$. □

Definition 4.2. Fix an integer $\alpha > 0$. Let $h_1 : Y_1 \to Y$ be a covering of degree ℓ. We say that a point P of Y is of *almost Galois type* $m_{h_1}(P) := k$ if $\ell \geq L_{1,\alpha}$ and $1 \leq k \leq 6$ is the minimal integer for which $E_{h_1}(P)$ contains at least $\ell/k - 2(\alpha+1)(k+1) - 2(\alpha+1)(k^2-1)/3$ entries that are equal to k, as in case (a) of Proposition 4.1. We say that P is of *almost Galois type* $m_{h_1}(P) := \infty$ if $E_{h_1}(P)$ has at most $2(\alpha+1)(k+1)$ entries equal to k (as in Proposition 4.1.(b)), and P is not of almost Galois type k for $1 \leq k \leq 6$, and $\ell \geq L_{1,\alpha}$.

For a point P of almost Galois type $m_{h_1}(P) = k < \infty$, we define the *error* $\varepsilon = \varepsilon_{\alpha,k}$ of h_1 at P to be $\varepsilon := 2(\alpha+1)k(k+1+(k^2-1)/3)$, so that ε bounds the sum of entries in $E_{h_1}(P)$ which are different from k.

Remark 4.3. (1) Under its assumptions, Proposition 4.1 also shows that $m_{h_1}(P) = m_{h_2}(P)$ for every point P of Y.

(2) We note that if P is a point of almost Galois type, the value $m_{h_1}(P)$ is independent of the constant α. Indeed, if $m_P := m_{h_1}(P)$ is finite, the requirement $\ell \geq L_{1,\alpha} \geq 812(\alpha+1)$ implies that $\ell > 2\varepsilon_{\alpha,m_P}$ and hence that the sum of entries equal to m_P in $E_{h_1}(P)$ is at least $\ell/2$. If $m_P = \infty$, the same requirement implies that the sum of entries which are at most 6 in $E_{h_1}(P)$ is less than $\ell/2$. Thus P cannot be a point of distinct almost Galois types for two values of α.

Corollary 4.4. *For every integer $\alpha > 0$, there exists a constant $L_{2,\alpha} \geq L_{1,\alpha}$ satisfying the following property. Let $h_i : Y_i \to Y$, $i = 1,2$ be coverings of degree $\ell \geq L_{2,\alpha}$, and $p : Z \to Y$ a minimal covering factoring through h_1 and h_2. Assume $g_Z < \alpha\ell$ and* $\deg p = \deg h_1 \cdot \deg h_2$, *so that every point P of Y is of almost Galois type by Proposition 4.1, for $i = 1,2$. Then (1) $g_Y \leq 1$; (2) the multiset $M_{h_1} := \{m_{h_1}(P) > 1 \,|\, P \in Y(\mathbb{K})\}$ is empty if $g_Y = 1$; and (3) M_{h_1} is one of the following if $g_Y = 0$:*

- ∞, ∞
- $\infty, 2, 2$
- $3, 3, 3$
- $2, 3, 6$
- $2, 4, 4$
- $2, 2, 2, 2$.

Moreover, (4) the number of branch points P of Y with $m_{h_1}(P) = 1$ is bounded by $2(\alpha+1)$, (5) there exists an integer N_α, depending only on α, such the number of points of Y_1 lying over P is at most N_α for every $P \in Y(\mathbb{K})$ with $m_{h_1}(P) = \infty$, and (6) $g_{Y_i} \leq \alpha+1$ for $i \in I$.

Remark 4.5. The proof relies on the proof of [9, Corollary 7.3]. Namely, repeating this proof gives a constant M_α, depending only on α, which satisfies the following property. If $h_1 : Y_1 \to Y$ is a covering of degree $\ell \geq M_\alpha$ such that every $P \in Y(\mathbb{K})$ is of almost Galois type $m_{h_1}(P)$ with constant α, and if (4) and (6) hold, then (1)-(3) and (5) hold as well.

Proof. The Riemann-Hurwitz formula for the natural projection $p_2 : Z \to Y_2$ gives:

$$2(\alpha+1)\ell - 2 > 2(g_Z - 1 + \ell) = \sum_{P \in Y(\mathbb{K})} R_{p_2}(h_1^{-1}(P)). \tag{4.6}$$

For a fixed point P of Y, we have

$$R_{p_2}(h_1^{-1}(P)) = \sum_{e \in E_{h_1}(P), f \in E_{h_2}(P)} (e - (e,f)),$$

by Abhyankar's lemma 2.2. We next bound $|R_{p_2}(h_1^{-1}(P)) - \ell R_{h_1}(P)|$ in terms of α, for a point P with $m_{h_1}(P) = 1$. Let $m_P := m_{h_1}(P)$ and let $\varepsilon_P := \varepsilon(\alpha, m_P)$ be the error at P, for a point P of Y. Since the number of entries of $E_{h_1}(P)$ which equal m_P is at least $(\ell - \varepsilon_P)/m_P$ for every point P, for $m_P = 1$ the sum

$$\sum_{\substack{e \in E_{h_1}(P), f \in E_{h_2}(P) \\ f=1}} (e - (e,f)) = |f \in E_{h_1}(P) \colon f = 1| \sum_{e \in E_{h_1}(P)} (e-1)$$

is at least $(\ell - \varepsilon_P)\sum_{e \in E_{h_1}(P)}(e-1)$ and at most $\ell \sum_{e \in E_{h_1}(P)}(e-1)$. Since the sum and hence number of entries different from m_P is at most ε_P, one also has

$$\sum_{\substack{e \in E_{h_1}(P), f \in E_{h_2}(P) \\ e \neq 1, f \neq 1}} (e - (e,f)) < \varepsilon_P(\varepsilon_P - 1),$$

for every point P of Y. Combining these inequalities for a point P with $m_P = 1$, we have:

$$\begin{aligned} \big|R_{p_2}(h_1^{-1}(P)) - \ell R_{h_1}(P)\big| &= \Big|\sum_{e \in E_{h_1}(P), f \in E_{h_2}(P)} \big(e - (e,f)\big) - \ell \sum_{e \in E_{h_1}(P)} (e-1)\Big| \\ &\leq \Big|\sum_{\substack{e \in E_{h_1}(P), f \in E_{h_2}(P) \\ f=1}} \big(e - (e,f)\big) - \ell \sum_{e \in E_{h_1}(P)} (e-1)\Big| \\ &\quad + \Big|\sum_{\substack{e \in E_{h_1}(P), f \in E_{h_2}(P) \\ e \neq 1, f \neq 1}} \big(e - (e,f)\big)\Big| \\ &< \varepsilon_P \sum_{e \in E_{h_1}(P)} (e-1) + \varepsilon_P(\varepsilon_P - 1) \leq 2\varepsilon_P(\varepsilon_P - 1) < \delta_\alpha. \end{aligned} \tag{4.7}$$

for some constant $\delta_\alpha > 1$, depending only on α. In particular, $R_{p_2}(h_1^{-1}(P)) \geq \ell - \delta_\alpha$ if P ramifies under h_1. Hence (4.6) and (4.7) give $2(\alpha+1)\ell > B(\ell - \delta_\alpha)$ where B is the number of branch points of h_1 with $m_P = 1$. For $\ell \geq \delta_\alpha(2\alpha+3)$, this implies that $B \leq 2(\alpha+1)$, proving part (4). Since $\ell(g_{Y_1} - 1) \leq g_Z - 1$ by the Riemann–Hurwitz formula and since by assumption $g_Z < \alpha\ell$, we also have $g_{Y_1} < \alpha + 1$, proving (6). Parts (1)-(3) and (5) then follow by Remark 4.5 for ℓ at least $L_{2,\alpha} := \max\{\delta_\alpha(2\alpha+3), M_\alpha\}$. □

Definition 4.6. Fix integers $\alpha > 0$ and $t \geq 2$. Let $h_i : Y_i \to Y$, $i = 1, \ldots, t$ be coverings of degree ℓ, every pair of which has irreducible fiber product $Y_{i,j}$. We say that h_i, $i = 1, \ldots, t$ *admit a pairwise genus bound* $\alpha\ell$ if $g_{Y_{i,j}} < \alpha\ell$, and $\ell \geq L_{2,\alpha}$.

Define the *total error* $\varepsilon = \varepsilon_\alpha$ as the sum of errors $2(\alpha+1)\varepsilon_{\alpha,1} + \varepsilon_{\alpha,2} + \varepsilon_{\alpha,3} + \varepsilon_{\alpha,6}$. We call the constant $N = N_\alpha$ in Corollary 4.4, the *entry bound*.

Note that for coverings h_i, $i = 1, \ldots, t$, as in the definition, Proposition 4.1 implies that every point P is of almost Galois type under h_i, $i = 1, \ldots, t$. Moreover, all of the consequences of Corollary 4.4 hold, in particular giving the possibilities for $m_{h_i}(P)$, $P \in Y(\mathbb{K})$, for every $i = 1, \ldots, t$.

The total error ε is chosen here to bound the sum of all errors over all branch points $P \in Y(\mathbb{K})$ of h_1 with finite $m_{h_1}(P)$, when M_{h_1} is as in Corollary 4.4. Note that indeed the proof of the corollary shows that the number of branch points P with $m_{h_1}(P) = 1$ is at most $2(\alpha+1)$.

Note that for coverings h_i, $i = 1, \ldots, t$, with pairwise genus bound $\alpha\ell$, the values $m_{h_i}(P)$ are independent of the constant α, and independent of i for every point P of Y, by Remark 4.3.

Lemma 4.7. *Let $t \geq 2$, and let $h_i : Y_i \to Y$, $i = 1, \ldots, t$ be coverings such that the fiber product of every two of which is irreducible. Suppose that h_i, $i = 1, \ldots, t$ admit a pairwise genus bound $\alpha\ell$. Assume that all entries $r \in E_{h_j}(P)$ divide $m_{h_j}(P)$ for all points P of Y with finite $m_{h_j}(P)$, for some $1 \leq j \leq t$. Then the monodromy group G_j of h_j is solvable of derived length at most 2.*

Proof. Let $\tilde{h}_j : \tilde{Y}_j \to Y$ be the Galois closure of h_j. We first claim that either $g_{\tilde{Y}_j} = 1$ or $g_{\tilde{Y}_j} = 0$ and G_j is cyclic or dihedral. The lemma then follows from the claim since the monodromy group of Galois coverings $\tilde{Y}_j \to Y$ of genus $g_{\tilde{Y}_j} = 1$ is well known to be a subgroup of a semidirect product of two abelian groups [26, Lemma 9.6].

By Corollary 4.4 we have $g_Y \leq 1$. Moreover in the case $g_Y = 1$, the corollary implies $m_{h_j}(P) = 1$ for all points P of Y. Thus in this case, h_j is unramified by our assumption. It follows that h_j is a morphism of elliptic curves and hence is Galois by [31, Theorem III.4.8], so that $\tilde{Y}_j = Y_j$ is of genus 1, proving the claim in this case. Henceforth assume $g_Y = 0$.

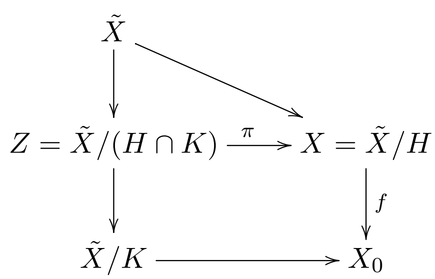
By Corollary 4.4, the mutiset $M_{h_j} := \{m_{h_j}(P) \,|\, P \in Y(\mathbb{K}), m_{h_j}(P) > 1\}$ is one of $\{\infty,\infty\}, \{\infty,2,2\}, \{2,2,2,2\}, \{3,3,3\}, \{2,4,4\}, \{2,3,6\}$. Put $u := \deg \tilde{h}_j$. Since the ramification index of $\tilde{h}_j$ over a point P is the least common multiple of all $r \in E_{h_j}(P)$ by Abhyankar's lemma (see Remark 2.3), our assumption implies that in the latter four cases, the ramification of $\tilde{h}_j$ is $[2^{u/2}]$ four times, or $[3^{u/3}]$ three times, or $[2^{u/2}], [4^{u/4}]$ twice, or $[2^{u/2}], [3^{u/3}], [6^{u/6}]$. Thus, the Riemann–Hurwitz formula for $\tilde{h}_j$ implies that $g_{\tilde{Y}_j} = 1$ in the latter four possibilities for M_{h_j}, proving the claim in these cases.

If $M_{h_j} = \{\infty,\infty\}$, our assumption implies that h_j has no branch points P with finite $m_{h_j}(P)$, and hence only two branch points in total. The only possibility for the ramification of a covering h_j which satisfies the Riemann–Hurwitz formula and has at most two branch points is $[\ell], [\ell]$, where $\ell := \deg h_j$. In this case, as in Section 2.4, G_j is generated by two elements with product 1, and hence $G_j \cong C_\ell$.

It remains to consider the case where $m_{h_j}(P_0) = \infty$, $m_{h_j}(P_1) = m_{h_j}(P_2) = 2$ for three points P_0, P_1, P_2 of Y, and $m_{h_j}(P) = 1$ for every other point $P \in Y(\mathbb{K})$. By assumption $r = 1$ for all $r \in E_{h_j}(P)$, $P \neq P_j$, $j = 0, 1, 2$ and $r = 1$ or 2 for $r \in E_{h_j}(P_j)$, $j = 1, 2$. Since h_j has such ramification, as in Section 2.4 there exists a product 1 tuple x_1, x_2, x_3 for G_j where x_2 and x_3 are elements of order 2. Since G_j is generated by the order two elements x_2 and x_3, the group G_j must be cyclic or dihedral, proving the claim. □

5. The genus of a product type covering

Fix $t \geq 2$, and consider indecomposable coverings $f : X \to X_0$ with Galois closure $\tilde{X}$, and monodromy group $G \leq S_\Delta \wr S_I$ such that $|I| = t$. Letting $H \leq G$ be a point stabilizer and $K := G \cap S_\Delta^I$, we assume $G = H \cdot K$, and put $\ell := |\Delta|$ and $m := [G : K]$.



The following proposition describes the Riemann–Hurwitz contribution of the natural projection $\pi : Z \to X$, where $Z := \tilde{X}/(H \cap K)$.

Proposition 5.1. *There exists a constant $E_0 = E_{0,t} > 0$, depending only on t, satisfying the following property. Let $f : X \to X_0$ and $\pi : Z \to X$ be coverings as above, and P a point of X_0. Let $y = a\sigma \in S_\Delta \wr S_I$, $a \in S_\Delta^I$, $\sigma \in S_I$, let O be the set of orbits of σ on I, and $E \subseteq O$ the subset of even length orbits. Assume y is a reduced form of a branch cycle x of f over P, with representatives $\iota_\theta, \theta \in O$.*

1. *Then $R_\pi(f^{-1}(P)) < m\ell^{t-1}/2 + E_0\ell^{t-2}$;*
2. *If $|E| \neq 1$ or $E = \{\mu\}$ with $|\mu| > 2$, then $R_\pi(f^{-1}(P)) < E_0\ell^{t-2}$;*
3. *If $t = 2$ and $\sigma \neq 1$, then $R_\pi(f^{-1}(P))$ is the number of odd orbits of $a(\iota_I)$.*

Proof. **Step I:** *Expressing $R_\pi(f^{-1}(P))$ in terms of orbits of y.* Let M be the G-set $K\backslash G$ so that $|M| = [G : K] = m$. Since $H \cap K$ (resp. H) is a point stabilizer in the action of G on $\Delta^I \times M$ (resp. Δ^I), the points of Z (resp. of X) lying over P are in one to one correspondence with the orbits of x acting on $(H \cap K)\backslash G \cong \Delta^I \times M$ (resp., $H\backslash G \cong \Delta^I$) as G-sets. Note that since $G = HK$, we have $\deg \pi = [H : H \cap K] = [G : K] = m$.

By the chain rule (2.1), $R_{f\circ\pi}(P) = mR_f(P) + R_\pi(f^{-1}(P))$. Hence, by the previous paragraph

$$(5.1)\quad \begin{aligned} R_\pi(f^{-1}(P)) = R_{f\circ\pi}(P) - mR_f(P) &= m\ell^t - |\mathrm{Orb}_{\Delta^I\times M}(x)| - m(\ell^t - |\mathrm{Orb}_{\Delta^I}(x)|) \\ &= m\cdot|\mathrm{Orb}_{\Delta^I}(x)| - |\mathrm{Orb}_{\Delta^I\times M}(x)|. \end{aligned}$$

Let $\kappa \in S_\Delta^I$ be such that $x^\kappa = y$. Note that y acts on $\hat{M} := S_\Delta^I\backslash(S_\Delta^I\cdot G)$. We claim that

$$(5.2)\quad R_\pi(f^{-1}(P)) = m\cdot|\mathrm{Orb}_{\Delta^I}(y)| - |\mathrm{Orb}_{\Delta^I\times\hat{M}}(y)|.$$

Let $\hat{G} := S_\Delta^I\cdot G$, and let $\hat{H}$ be a point stabilizer of $\hat{G}$ such that $\hat{H}\cap G = H$. As x and y are conjugate in $\hat{G}$, we have $|\mathrm{Orb}_{\hat{H}\backslash\hat{G}}(x)| = |\mathrm{Orb}_{\hat{H}\backslash\hat{G}}(y)|$, as well as $|\mathrm{Orb}_{(\hat{H}\backslash\hat{G})\times\hat{M}}(x)| = |\mathrm{Orb}_{(\hat{H}\backslash\hat{G})\times\hat{M}}(y)|$. Since $\hat{H}\backslash\hat{G}\cong\Delta^I$ and $(\hat{H}\backslash\hat{G})\times\hat{M}\cong\Delta^I\times\hat{M}$ as $\hat{G}$-sets, and $\hat{M}\cong M$ as G-sets, these equalities give

$$|\mathrm{Orb}_{\Delta^I}(x)| = |\mathrm{Orb}_{\Delta^I}(y)| \text{ and } |\mathrm{Orb}_{\Delta^I\times M}(x)| = |\mathrm{Orb}_{\Delta^I\times\hat{M}}(x)| = |\mathrm{Orb}_{\Delta^I\times\hat{M}}(y)|.$$

Plugging these equalities into (5.1), gives the claim (5.2).

Step II: *Computing the length of an orbit in the action of y on $\Delta^I\times\hat{M}$.* Denote by $\mathcal{R}$ the set of all $R = \prod_{i\in I} R_i$ such that R_i is an orbit of $a(\iota_\theta)$, for all $i\in\theta$, $\theta\in O$. Throughout the proof, $R_i\subseteq\Delta$ denotes the projection $R\in\mathcal{R}$ to the i-th coordinate. Also put $r_i := |R_i|$, for every $R\in\mathcal{R}, i\in I$. Moreover for $\theta\in O$, $\delta\in R$ and $R\in\mathcal{R}$, denote by δ_θ the tuple $(\delta(i))_{i\in\theta}$, let $R_\theta := \{\delta_\theta : \delta\in R\}$, and let $r_\theta := \mathrm{lcm}_{i\in\theta}(r_i)$ be the length of the orbits of $a(\iota_\theta)$ on R_θ.

First note that as $\hat{M} = S_\Delta^I\backslash(S_\Delta^I\cdot G)$, the action of y on $\hat{M}$ is through its image σ. As the latter acts regularely, the orbits of y on $\hat{M}$ are of length $e := |\langle\sigma\rangle| = \mathrm{lcm}_{\theta\in O}(|\theta|)$. Fix $R\in\mathcal{R}$ and $\theta\in O$. Let $\hat{k}_\theta$ denote the length of the orbit of $(\delta_\theta, m)\in R_\theta\times\hat{M}$ under y. Since $|\theta|$ divides e, it also divides $\hat{k}_\theta$. Write $z := aa^{\sigma^{-1}}\cdots a^{\sigma^{1-|\theta|}}$, so that $y^{\hat{k}_\theta} = (z\sigma^{|\theta|})^{\hat{k}_\theta/|\theta|}$. It follows that $\hat{k}_\theta$ is the smallest multiple of $|\theta|$ for which $y^{\hat{k}_\theta} = (z\sigma^{|\theta|})^{\hat{k}_\theta/|\theta|}$ acts as the identity on R_θ, or equivalently $z^{\hat{k}_\theta/|\theta|}$ acts as the identity on R_θ. Since $z(i) = a(\iota_\theta)$ for $i\in\theta$, and each orbit of $a(\iota_\theta)$ on R_θ has length r_θ, we see that the tuple $z_\theta = (z(i))_{i\in\theta}\in R_\theta$ is of order r_θ. Hence $\hat{k}_\theta = \mathrm{lcm}(e, |\theta|\cdot r_\theta)$.

The length $\hat{k}$ of the orbit of $(\delta, m)\in R\times\hat{M}$ is the least common multiple of the lengths $\hat{k}_\theta$ of the orbits of $(\delta_\theta, m), \theta\in O$. As $e = \mathrm{lcm}_{\theta\in O}(|\theta|)$, we get

$$\hat{k} = \mathrm{lcm}_{\theta\in O}(\hat{k}_\theta) = \mathrm{lcm}_{\theta\in O}(\mathrm{lcm}(e, |\theta|\cdot r_\theta)) = \mathrm{lcm}_{\theta\in O}(|\theta|\cdot r_\theta).$$

Step III: *Describing the length of an orbit on Δ^θ, for $\theta\in O$.* Fix $R\in\mathcal{R}$ and $\theta\in O$. Let k denote the length of some orbit of y on R_θ. Since $y^{|\theta|r_\theta}$ acts trivially on R_θ by step II, we have $k \mid |\theta|\cdot r_\theta$. As in step II, the orbits of the element $y^{|\theta|} = z\sigma^{|\theta|}$ on R_θ are of length r_θ, and hence $k = |\theta|\cdot r_\theta/v_\theta$ for some integer v_θ which divides $|\theta|$ and is prime[2] to r_θ.

[2]This statement follows from the following elementary fact: if x^u is an element of order v then x is an element of order uv/w for some w dividing u which is prime to v.

The length of the orbit of $\delta \in R_\theta$ is then k if and only if k is minimal positive integer such that $\delta(i) = \delta^{y^k}(i) = \delta^{\sigma^k b}(i)$ for all $i \in \theta$, where $b := \sigma^{-k}y^k \in S_\Delta^\theta$, or equivalently

$$\delta(i^{\sigma^{-k}}) = \delta^{\sigma^k}(i) = \delta^{y^k b^{-1}}(i) = \delta^{b^{-1}}(i) = \delta(i)^{b^{-1}(i)} \text{ for all } i \in \theta. \tag{5.3}$$

Since $(r_\theta, v_\theta) = 1$, and $k = |\theta| r_\theta / v_\theta$, and since $\sigma^{|\theta|/v_\theta}$ has orbits of length v_θ on θ, the elements σ^{-k} and $\sigma^{|\theta|/v_\theta}$ have the same orbits on θ. If $\delta \in R_\theta$ has an orbit length k, it follows from (5.3) that its values on the orbit of i under $\sigma^{|\theta|/v_\theta}$ are determined by a single value $\delta(i)$, for all $i \in \theta$.

Step IV: *Estimating the number of elements in Δ^I with a fixed orbit length.* Let V be the set of tuples $v = (v_\theta)_{\theta \in O}$, where $v_\theta \mid |\theta|$ is a positive integer for all $\theta \in O$. For $v \in V$, let Ω_v be the set of all $\delta \in \Delta^I$ such that the length of the orbit of δ_θ is $|\theta| \cdot r_\theta / v_\theta$, for all $\theta \in O$. By the claim in step III, Δ^I is a (disjoint) union of the sets Ω_v, $v \in V$.

Since an element $\delta \in \Omega_v$ satisfies (5.3) with $k = |\theta| \cdot r_\theta / v_\theta$ for each $\theta \in O$, the values of δ on each orbit $\theta' \subseteq \theta$ of $\sigma^{|\theta|/v_\theta}$ are determined by its value on a single representative $\iota_{\theta'} \in \theta'$. Since there are $|\theta|/v_\theta$ such orbits θ' and at most ℓ possible values $\delta(\iota_{\theta'})$ for each representative, we get the estimate $|\Omega_v| \leq \ell^{\sum_{\theta \in O} |\theta|/v_\theta}$ for $v \in V$.

Step V: *Isolating the main term of $R_\pi(f^{-1}(P))$ and deriving parts (1)-(2).* For $v \in V$, let $\mathcal{R}_v$ denote the set of tuples $R \in \mathcal{R}$ such that $\Omega_v \cap R$ is nonempty.

The element y acts on the set $\Omega_v \cap R$ and its orbits in this action are all of length $k_{v,R} := \mathrm{lcm}_{\theta \in O}(|\theta| \cdot r_\theta / v_\theta)$, for $v \in V, R \in \mathcal{R}_v$. Similarly by step II, the length of each orbit on $(\Omega_v \cap R) \times \hat{M}$ is $\hat{k}_{v,R} := \mathrm{lcm}_{\theta \in O}(|\theta| \cdot r_\theta)$. In particular, y has $|\Omega_v \cap R| / k_{v,R}$ orbits on $\Omega_v \cap R$ and $|\Omega_v \cap R| \cdot m / \hat{k}_{v,R}$ orbits on $(\Omega_v \cap R) \times \hat{M}$. Thus (5.2) gives

$$R_\pi(f^{-1}(P)) = m \sum_{v \in V} T_v \text{ where } T_v := \sum_{R \in \mathcal{R}_v} \Big(\frac{1}{k_{v,R}} - \frac{1}{\hat{k}_{v,R}} \Big) |\Omega_v \cap R|. \tag{5.4}$$

Since $T_v \leq |\Omega_v| \leq \ell^{\sum_{\theta \in O} |\theta|/v_\theta}$ by step IV, we have $T_v \leq \ell^{t-2}$ for every $v \in V$ satisfying $\sum_{\theta \in O} |\theta|/v_\theta \leq t-2$. Moreover, if $\sum_{\theta \in O} |\theta|/v_\theta = t$ then $v_\theta = 1$ for all $\theta \in O$, implying that $k_{v,R} = \hat{k}_{v,R}$ for all $R \in \mathcal{R}_v$, and hence $T_v = 0$. Let U be the set of tuples $v \in V$ such that $\sum_{\theta \in O} |\theta|/v_\theta = t-1$. Since $|V|$ is bounded by a constant $E_0' = E_{0,t}'$ depending only on t, and $T_v \leq \ell^{t-2}$ for $v \in V \setminus U$, (5.4) gives:

$$R_\pi(f^{-1}(P)) < m \sum_{v \in U} T_v + m E_0' \ell^{t-2} \leq m \sum_{v \in U} T_v + E_0 \ell^{t-2}, \tag{5.5}$$

where $E_0 := t! E_0'$, and the latter inequality follows since $m \leq t!$ as G/K is a subgroup of S_t.

It remains to estimate $|U|$. Note that as $t = \sum_{\theta \in O} |\theta|$, one has $v \in U$ if and only if $\sum_{\theta \in O} (|\theta| - |\theta|/v_\theta) = 1$. Hence $v \in U$ if and only if $v_\theta = 1$ for all but a single orbit $\mu \in O$ such that $v_\mu = 2$ and $|\mu| = 2$. In this case, $(r_\mu, 2) = (r_\mu, v_\mu) = 1$ for all $R \in \mathcal{R}_v$ by step III. As $k_{v,R} = \mathrm{lcm}_{\theta \in O}(r_\theta \cdot |\theta|/v_\theta)$, and $\hat{k}_{v,R} = \mathrm{lcm}_{\theta \in O}(r_\theta \cdot |\theta|)$ for $v \in U$, we have

$$k_{v,R} = \hat{k}_{v,R} \text{ if } |\theta| \cdot r_\theta \text{ is even for some } \theta \in O \setminus \{\mu\}, \text{ and } \hat{k}_{v,R} = 2k_{v,R} \text{ otherwise,} \tag{5.6}$$

for all $R \in \mathcal{R}_v$. In particular, if $|E| > 1$ or $E = \{\mu\}$ with $|\mu| \neq 2$, then $T_v = 0$ for all $v \in U$, so that (5.5) gives part (2). Moreover, if $E = \{\mu\}$ with $|\mu| = 2$, we have $U = \{v\}$, where $v_\mu = 2$ and $v_\theta = 1$ for all $\theta \in O \setminus \{\mu\}$. Since $|\Omega_v| \leq \ell^{t-1}$ by step IV, and $1/k_{v,R} - 1/\hat{k}_{v,R} \leq 1/2$ by (5.6), we deduce from (5.5) that

$$R_\pi(f^{-1}(P)) < mT_v + E_0\ell^{t-2} \leq \frac{m}{2}\sum_{R\in\mathcal{R}_v} |\Omega \cap R| + E_0\ell^{t-2} \leq \frac{m\ell^{t-1}}{2} + E_0\ell^{t-2},$$

proving part (1).

Step VI: *Proving part (3).* Assume $t = 2$ and $\sigma \neq 1$. We compute $R_\pi(f^{-1}(P))$ using (5.4). Note that $I = \mu$ is the only orbit of σ. Write $I = \{1, 2\}$ with $1 = \iota_\mu$, and $y = (a(1), 1)\sigma$ in the notation of Section 2.5. Note that $V = \{u, v\}$ where $v_\mu = 2$ and $u_\mu = 1$. As in step V, $T_u = 0$, so that it remains to find T_v.

We first claim that for every $R \in \mathcal{R}_v$ and $\delta \in \Omega_v \cap R$, one has $R_1 = R_2$ with odd r_1, and $\delta(2) = \delta(1)^{b^{-1}(1)}$ for $b = \sigma y^{r_1}$. By step III, r_μ and hence r_1 are odd. By (5.3), $\delta(2) = \delta(1)^{b^{-1}(1)}$, where $b := \sigma^{-r_\mu} y^{r_\mu} = \sigma y^{r_\mu} \in S_\Delta^I$. Since $y = (a(1), 1)\sigma$, one has $b(1) \in \langle a(1)\rangle$, and hence $\delta(2) \in R_1$. Thus, $R_1 = R_2$, and $r_\mu = r_1$ is odd, proving the claim. In this case we have $k_{v,R} = r_1$ and $\hat{k}_{v,R} = 2r_1$.

Conversely, we claim that for an odd length orbit R_1 of $a(1)$, and $\alpha \in R_1$, there exists a unique $\delta \in R_1^I \cap \Omega_v$ such that $\delta(1) = \alpha$. Indeed, letting $\delta := (\alpha, \alpha^{b(1)}) \in R_1^I$ where $b := y^{r_1}\sigma$, a direct computation shows that the orbit of δ is of length r_1, and hence $\delta \in \Omega_v \cap R_1^I$. The uniqueness of δ follows from (5.3), proving the claim.

The two claims in this step imply that $|\Omega_v \cap R|$ is r_1 if $R_1 = R_2$ and r_1 is odd, and is 0 otherwise. Since in the former case we got $k_{v,R} = r_1$ and $\hat{k}_{v,R} = 2r_1$, we deduce that T_v is half the number of odd length orbits of $a(1)$. Since $R_\pi(f^{-1}(P)) = mT_v = 2T_v$ by (5.4), this proves part (3). □

We will need the following further bound on the Riemann–Hurwitz contribution R_π when the entries $a(\iota_\theta)$ of the reduced form are of almost Galois type. For this we shall use the following definition for a permutation to be almost Galois:

Definition 5.2. We say that $x \in S_\ell$ is *a permutation of almost Galois type* $m < \infty$ (resp., $m = \infty$) with *error* at most ε (resp., *entry bound* N) if the number of length m orbits of x is at least $(\ell - \varepsilon)/m$ (resp., the number of orbits of x is at most N).

Thus, the error bounds the sum of the lengths r of orbits of x with $r \neq m$.

Corollary 5.3. *For every $\varepsilon > 0$, there exists a constant $E_1 = E_{1,\varepsilon,t}$, depending only on ε and t with the following property. Let $f : X \to X_0$ and $\pi : Z \to X$ be coverings as above, and $y = a\sigma \in S_\Delta \wr S_I$ be a reduced form of a branch cycle of f over a point P, where $a \in S_\Delta^I$, $\sigma \in S_I$. Let ι_θ denote the representative of y for an orbit $\theta \subseteq I$ of σ. If $a(\iota_\theta)$ is of almost Galois type with error at most ε for every orbit $\theta \subseteq I$ of σ, and $m(a(\iota_\eta))$ is infinite or even for some orbit $\eta \subseteq I$ of σ, then $R_\pi(f^{-1}(P)) < E_1\ell^{t-2}$.*

Proof. Let $\eta \in O$ be an orbit for which $a(\iota_\eta)$ is of almost Galois type with error at most ε, such that $m(a(\iota_\eta))$ is even or infinite. We use the estimates and notation of the proof of Proposition 5.1. Let O be the set of orbits of σ on I. By choosing $E_1 \geq E_0$ and applying Proposition 5.1.(3), we reduce to the case where σ has a single orbit $\mu \in O$ of even length, and this orbit is of length $|\mu| = 2$. As in Step V, $R_\pi(f^{-1}(P))$ breaks into the sum $m\sum_{v\in V} T_v$, and as in (5.5), it suffices to estimate T_v for the unique $v \in U$, that is, for the tuple $v = (v_\theta)_{\theta\in O}$ where $v_\theta = 1$ for all $\theta \in O \setminus \{\mu\}$ and $v_\mu = 2$.

As in Proposition 5.1, let $\mathcal{R}$ be the set of all $R = \prod_{i\in I} R_i$, where R_i is an orbit $a(\iota_\theta)$ for all $i \in \theta$, $\theta \in O$, let $r_i := \#R_i$ and let $r_\theta := \mathrm{lcm}_{i\in\theta}(\#R_i)$ for $\theta \in O$. Also let Ω_v be the set of all $\delta \in \Delta^I$ such that the length of the orbit of $\delta_\theta := (\delta(i))_{i\in\theta}$ is $|\theta| \cdot r_\theta/v_\theta$, for all $\theta \in O$, so that the length $k_{v,R} := \mathrm{lcm}_{\theta\in O}(r_\theta \cdot |\theta|/v_\theta)$ of an orbit of $\delta \in \Omega_v \cap R$ is smaller than the length $\hat{k}_{v,R} := \mathrm{lcm}_{\theta\in O}(r_\theta \cdot |\theta|)$ of orbits in $R \times (G/K)$. By step III of the proposition, if $\Omega_v \cap R \neq \emptyset$ then r_μ is odd. Moreover by (5.6), $T_v = 0$ if r_i is even for some $i \in I \setminus \mu$. Henceforce assume all $r_i, i \in I$ are odd, so that (5.6) gives $\hat{k}_{v,R} = 2k_{v,R} \geq r_{\iota_\eta}$. Thus,

$$T_v = \sum_{R\in\mathcal{R}} \Big(\frac{1}{k_{v,R}} - \frac{1}{\hat{k}_{v,R}}\Big)|\Omega_v \cap R| = \sum_{R\in\mathcal{R}} \frac{|\Omega_v \cap R|}{\hat{k}_{v,R}} \leq \sum_{R\in\mathcal{R}_v} \frac{|\Omega_v \cap R|}{r_{\iota_\eta}}.$$

Write $\mu = \{\iota_\mu, \iota_\mu'\}$. Since the values of $\delta \in \Omega_v \cap R$ on μ are determined by its value on ι_μ by (5.3), the latter inequality gives

$$T_v \leq \sum_{(R_i)_{i\in I\setminus\{\iota_\mu'\}}} \frac{r_{\iota_\mu}\prod_{i\in I\setminus\mu} r_i}{r_{\iota_\eta}}, \tag{5.7}$$

where R_i runs through odd length orbits of $a(\iota_\theta)$ for every $i \in \theta \setminus \{\iota_\mu'\}$. Cancelling out r_{ι_η} with r_{ι_μ} if $\mu = \eta$ and with r_i for $i = \iota_\eta \notin \mu$ otherwise, (5.7) gives:

$$T_v \leq \big|\{R_{\iota_\eta} \in \mathrm{Orb}_\Delta(a(\iota_\eta)) : r_{\iota_\eta} \text{ is odd}\}\big| \prod_{i\in I\setminus\{\iota_\mu',\iota_\eta\}} \sum_{R_i\in\mathrm{Orb}_\Delta(a(\iota_i))} r_i \tag{5.8}$$

Since $a(\iota_\eta)$ is of even or infinite almost Galois type, there is an upper bound $E_1' = E_{1,\varepsilon,t}'$, depending only on ε and t, on the number of odd length orbits of $a(\iota_\eta)$. Thus, as the sum over orbits of an element in S_Δ is at most ℓ, (5.8) gives:

$$T_v \leq E_1' \prod_{i\in I\setminus\{\iota_\mu',\iota_\eta\}} \sum_{R_i\in\mathrm{orb}_\Delta(a(\iota_i))} r_i \leq E_1'\ell^{t-2}. \tag{5.9}$$

Setting $E_1 := E_{1,t} := t!E_1' + E_0$ and noting that $m \leq t!$ as $G/K \leq S_t$, (5.5) and (5.9) give:

$$R_\pi(f^{-1}(P)) < mT_v + E_0\ell^{t-2} \leq (t!E_1' + E_0)\ell^{t-2} = E_1\ell^{t-2}.$$

□

6. Ramification of genus ≤ 1 with a transposition

Fix $t \geq 2$ and consider indecomposable coverings $f : X \to X_0$ with monodromy group $G \leq S_\Delta \wr S_I$ with $|I| = t$. Letting $K := G \cap S_\Delta^I$ and H a point stabilizer, as in Section 5 we assume $G = H \cdot K$. Assume further that the image $\overline{G} \cong G/K$ of G in S_I is transitive on I.

deg $\overline{\pi}_0$	*Ramification for* $\overline{\pi}_0$	*Ramification for* π_0	$\mathrm{Mon}(\overline{\pi}_0)$	g_Y
2	$[2]^2$	$[2]^2$	C_2	0
2	$[2]^4$	$[2]^4$	C_2	1
3	$[3], [2,1]^2$	$[3^2], [2^3]^2$	D_6	0
3	$[2,1]^4$	$[2^3]^4$	D_6	1
4	$[4], [2^2], [2,1^2]$	$[4^2], [2^4]^3$	D_8	0
4	$[2,1^2]^2, [2^2]^2$	$[2^4]^4$	D_8	1
4	$[4], [3,1], [2,1^2]$	$[4^6], [3^8], [2^{12}]$	S_4	0
6	$[6], [3,3], [2,1^4]$	$[6^4], [3^8], [2^{12}]$	$C_2 \times AGL(1,4)$	1

TABLE 6.1. Ramification types containing $[2, 1^{t-2}]$, for coverings $\overline{\pi}_0$ of $\mathbb{P}^1$ with Galois closure Y of genus $g_Y \le 1$.

The following proposition gives an upper bound on the total Riemann–Hurwitz contribution of the natural projection $\pi : Z \to X$, where Z is the quotient by $H \cap K$. This bound is then used to prove that the ramification is of almost Galois type, as described in Section 1. Let Y be the quotient by K, put $\ell := |\Delta|$ and $m := [G : K]$, and let E_0 be the constant from Proposition 5.1.

Proposition 6.1. *Assume $g_Y \le 1$ and let s be the number of branch points of f whose branch cycle acts on I as a transposition. Let $R_\pi := \sum_P R_\pi(f^{-1}(P))$ where P runs over all points of X_0. Then $R_\pi \le sm\ell^{t-1}/2 + 4E_0\ell^{t-2}$ if $s > 0$ and $t \ge 3$; and $R_\pi \le 2tE_0\ell^{t-2}$ if $s = 0$ and $t \ge 3$; and $R_\pi \le s\ell$ if $t = 2$. Moreover, $s \le 2$ if $t \ge 3$, and $s \le 4$ if $t = 2$.*

The proof follows from Proposition 5.1 and the following lemma:

Lemma 6.2. *Let $\overline{\pi}_0 : \overline{Y} \to \mathbb{P}^1$ be a covering with Galois closure of genus at most 1. Assume that the ramification type of $\overline{\pi}_0$ contains a multiset of the form $[2, k_1, \dots, k_u]$ with odd k_i for $i = 1, \dots, u$. Then $k_i = 1$ for $i = 1, \dots, u$, and the ramification type of $\overline{\pi}_0$ appears in Table 6.1.*

Proof of Proposition 6.1. Letting $\overline{H}$ be a point stabilizer in the action of G on I, we let $\pi_0 : Y \to X_0$ and $\overline{\pi}_0 : \overline{Y} \to X_0$ be the natural projections, where $\overline{Y} := \tilde{X}/\overline{H}$. Since G acts transitively on I, the action of $\overline{G} := \mathrm{Mon}(\overline{\pi}_0)$ on $G/\overline{H}$ is equivalent to its action on I. Since K is the kernel of this action, $\overline{G} \cong G/K$, and hence π_0 is the Galois closure of $\overline{\pi}_0$. In particular, π_0 and $\overline{\pi}_0$ have the same branch points by Remark 2.3.

Let S be the subset of branch points of $\overline{\pi}_0$ whose ramification type contains a tuple $[2, k_1, \dots, k_r]$ for some odd $k_1, \dots, k_r$, $r \in \mathbb{N}$. By Lemma 6.2, the number of branch points of π_0 is at most 4, and $|S| \le 2$ for $t \ge 3$ (resp. $|S| \le 4$ for $t = 2$). Moreover, Lemma 6.2 shows that S consists of points p in X_0 such that $E_{\overline{\pi}_0}(p) = [2, 1^{t-2}]$. Since $E_{\overline{\pi}_0}(p)$ equals the action of $\overline{G}$ on $\overline{H}\backslash\overline{G}$ is equivalent to its action on I, it follows that $s = |S|$. The desired bounds on s follow.

By Abhyankar's lemma 2.2, if p is not a branch point of π_0 then π is unramified over every preimage in $f^{-1}(p)$ and hence $R_\pi(f^{-1}(p)) = 0$. For a branch point $p \notin S$ of π_0, one

(A) $[m], [m]$ with $\overline{G} \cong C_m$;	(F) $[2^{m/2}]^4$;
(B) $[m/2, m/2], [2^{m/2}], [2^{m/2}]$ with $\overline{G} \cong D_m$;	(G) $[2^{m/2}], [4^{m/4}]^2$;
(C) $[2^6], [3^4], [3^4]$ with $\overline{G} \cong A_4$;	(H) $[2^{m/2}], [3^{m/3}], [6^{m/6}]$;
(D) $[2^{12}], [3^8], [4^6]$ with $\overline{G} \cong S_4$;	(I) $[3^{m/3}]^3$.
(E) $[2^{30}], [3^{20}], [5^{12}]$ with $\overline{G} \cong A_5$;	

Table 6.2. Ramification types for Galois coverings $Y \to \mathbb{P}^1$ of genus $g_Y \leq 1$.

has $R_\pi(f^{-1}(p)) < E_0\ell^{t-2}$ if $t \geq 3$ by Proposition 5.1.(2). For $p \in S$, one has $R_\pi(p) \leq m\ell^{t-1}/2 + E_0\ell^{t-2}$ if $t \geq 3$, and $R_\pi(f^{-1}(p)) \leq \ell$ if $t = 2$, by Proposition 5.1, parts (1) and (3). Since $g_Y \leq 1$, the Riemann–Hurwitz formula implies that $\overline{\pi}_0$ and hence π_0 have at most $2t$ branch points. Thus, we have $R_\pi \leq 2tE_0\ell^{t-2}$ if $s = 0$. Now assume $s \geq 1$, so that π_0 has at most four branch points by Lemma 6.2. The bounds on $R_\pi(f^{-1}(p))$ in this paragraph then give $R_\pi \leq ms\ell^{t-1}/2 + 4E_0\ell^{t-2}$ for $t \geq 3$ and $R_\pi \leq s\ell$ for $t = 2$. □

Proof of Lemma 6.2. Let $\pi_0 : Y \to \mathbb{P}^1$ denote the Galois closure of $\overline{\pi}_0$, so that both maps have the same branch points $p_1, \ldots, p_r$. Assume $p_1, \ldots, p_{r-1}$ are ordered by decreasing Riemann-Hurwitz contributions $R_{\overline{\pi}_0}(p_1) \geq \cdots \geq R_{\overline{\pi}_0}(p_{r-1})$, and that $E_{\overline{\pi}_0}(p_r) = [2, k_1, \ldots, k_r]$. Let $\overline{G}$ be the monodromy group of $\overline{\pi}_0$ equipped with an action on a set I of cardinality $t := \deg \overline{\pi}_0$.

We first claim that either $g_{\overline{Y}} = 0$ or the ramification of $\overline{\pi}_0$ is [2] four times. Clearly, $g_{\overline{Y}} \leq g_Y$ and $g_Y \leq 1$ by hypothesis. If $t > 2$ and $g_{\overline{Y}} = 1$ then the natural projection $Y \to \overline{Y}$ is unramified, and hence $[2^{m/t}, k_1^{m/t}, \ldots, k_r^{m/t}]$ appears in the ramification type of π_0, contradicting the fact that π_0 is Galois. If $t = 2$ and $g_{\overline{Y}} = 1$, then $\overline{\pi}_0$ is Galois, and the ramification of $\overline{\pi}_0$ is $[2]^4$, proving the claim.

Henceforth assume $g_{\overline{Y}} = 0$. As π_0 is the Galois closure of $\overline{\pi}_0$, Abhyankar's lemma (Remark 2.3) implies that

(6.1) $\quad e_{\pi_0}(p_i)$ is the least common multiple of entries in $E_{\overline{\pi}_0}(p_i)$, for $i = 1, \ldots, r$.

The ramification and monodromy groups of Galois coverings $\pi_0 : Y \to \mathbb{P}^1$ with genus $g_Y \leq 1$ is well known (see [17, Proposition 2.4] and proof of [26, Proposition 9.5]) and appears in Table 6.2. Moreover, $\overline{G}$ is solvable in cases (F)-(I). Running over each of the cases (A)-(I), (6.1) implies that either $k_i = 1$ for all i, or we are in case (H), and $e_{\pi_0}(p_r) = 6$, and $k_i = 1$ or 3 for all i. We separate case (H) into case (H1) $e_{\pi_0}(p_r) = 2$, and case (H2) $e_{\pi_0}(p_r) = 6$. In particular, as $e_{\pi_0}(p_r)$ is 2 or 6, case (I) does not occur.

Since $R_{\pi_0}(p_i) \geq R_{\overline{\pi}_0}(p_i) \cdot m/t$ by the chain rule (2.1), and $R_{\pi_0}(p_i) = m(1 - 1/e_{\pi_0}(p_i))$, we have

$$R_{\overline{\pi}_0}(p_i) \leq \frac{t}{m} R_{\pi_0}(p_i) = \frac{e_{\pi_0}(p_i) - 1}{e_{\pi_0}(p_i)} \cdot t \text{ for } i = 1, \ldots, r-1.$$

As $g_{\overline{Y}} = 0$, combining the latter inequality with the Riemann-Hurwitz formula for $\overline{\pi}_0$ gives:

$$\sum_{i=1}^{r-1} \frac{e_{\pi_0}(p_i) - 1}{e_{\pi_0}(p_i)} \geq \frac{1}{t} \sum_{i=1}^{r-1} R_{\overline{\pi}_0}(p_i) = \frac{2t - 2 - R_{\overline{\pi}_0}(p_r)}{t} = 2 - \frac{3}{t}, \tag{6.2}$$

in cases (A)-(G) and case (H1), and similarly,

$$\sum_{i=1}^{r-1} \frac{e_{\pi_0}(p_i) - 1}{e_{\pi_0}(p_i)} \geq \frac{2t - 2 - R_{\overline{\pi}_0}(p_r)}{t} \geq \frac{2t - 2 - 1 - (2/3)(t-2)}{t}, \tag{6.3}$$

in case (H2). Applying (6.2) gives $t = 2$ in case (A); $t < 6$ in case (B) with $m \geq 20$ if $t = 5$; $t < 5$ in case (C); $t < 6$ in case (D); $t < 6$ in case (E); $t \leq 4$ in case (H1); and $t \leq 6$ in cases (F) and (G), where the latter is an equality if and only if $E_{\overline{\pi}_0}(p_i) = [e^{t/e}]$ for $e = e_{\pi_0}(p_i)$, $i < r$. Similarly (6.3) gives $t \leq 10$ in case (H2). Moreover if $t = 10$ in the latter case, then $E_{\overline{\pi}_0}(p_r) = [2, 3^{(t-2)/3}]$ and $E_{\overline{\pi}_0}(p_i) = [3^{t/3}]$ for some $i < r$. As both equalities cannot hold simultaneously with integral exponents, we can assume $t \leq 9$.

Since $\overline{G}$ acts transitively and faithfully on I, these conditions force the regular action with $t = m = 2$ if $\overline{G}$ is cyclic (case (A)); the standard action of D_{2t} on t elements with $m = 2t = 6$ or 8 if $\overline{G}$ is dihedral (case (B)); the standard action on a set of size $t = 4$ if $\overline{G} \cong A_4$ or S_4 (cases (C)-(D)); and the standard action on a set of size $t = 5$ if $\overline{G} \cong A_5$ (case (E)). Since the groups A_4 and A_5 do not contain transpositions in their standard action, we also have $\overline{G} \neq A_4, A_5$.

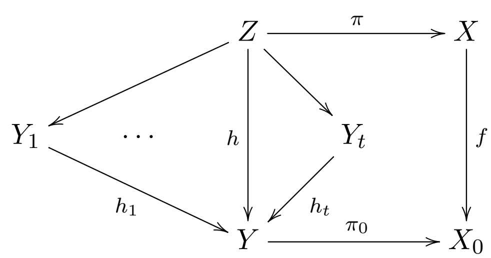
It follows that the ramification of π_0 is $[2]$ twice, or $[t,t], [2^t]$ twice with $t = 3$ or 4, or $[2^{12}], [3^8], [4^6]$, or as in cases (F)-(H). Given the bounds on t in the above paragraph and these possibilities for the ramification of π_0, a straightforward computation determines which ramification data for $\overline{\pi}_0$ contains $[2, 1^{t-2}]$ in cases (A)-(G) and (H1) (resp. $[2, 3^{u_1}, 1^{v_1}]$ for some integers $u_1 \geq 1$, $v_1 \geq 0$ in case (H2)), satisfies (6.1), and the Riemann–Hurwitz formula for $\overline{\pi}_0$. These are the ramification types in Table 6.1 and the two ramification types (F.N1) $[2, 1^4], [2^3]$ three times, and (H2.N1) $[2^3], [3^2], [1, 2, 3]$. Using multiplicativity of ramification indices, it is straightforward to check that every covering with ramification index F.N1 or H2.N1 is indecomposable. There is no indecomposable covering with ramification type F.N1 by Lemma 2.4.(a) as $\overline{\pi}_0$ is non-Galois. Since A_4 contains no element of order 6 and $\overline{G}$ contains a branch cycle of order 6 over p_r in case H2, $\overline{G}$ is not a quotient A_4, and hence there is no indecomposable covering with ramification H2.N1 by Lemma 2.4.(c).

For the ramification types with $g_Y = 0$, the corresponding group $\overline{G}$ is read off cases (A)-(E) above. For the ramification types with $g_Y = 1$ and $t \leq 4$, the subgroup $\overline{G} \leq S_t$, $t = 3, 4$, is of derived length at most 2, is transitive, and is not contained in A_t, forcing $\overline{G} \cong D_{2t}$. In case the ramification of $\overline{\pi}_0$ is $[6], [3, 3], [2, 1^4]$, as in Section 2.4 its monodromy group $\overline{G} \leq S_6$ is a transitive subgroup generated by a product 1 tuple $x_1, x_2, x_3 \in S_6$ with cycle structures $[6], [3, 3], [2, 1^4]$, respectively. A direct computation with Magma shows that for such tuples $\overline{G} = \langle x_1, x_2, x_3 \rangle \cong C_2 \times \mathrm{AGL}(1, 4)$. □

7. Theorem 1.1: reduction to almost Galois types and the case $t \geq 3$

In this section we deduce Theorem 1.1 from Theorem 7.1, which deals with the case of coverings with a pairwise genus bound, which by Section 4 ensures ramification of almost Galois type. Moreover, we prove Theorem 7.1 for groups of product type $G \leq S_\Delta \wr S_I$ with $|I| \geq 3$ in the absence of branch cycles that act on I as a transposition.

We assume the setup of Section 2.7, so that to a covering $f : X \to X_0$ with monodromy $G \leq S_\Delta \wr S_I$ of product type and Galois closure $\tilde{X}$, one associates the natural projections $\pi : Z \to X$, $h : Z \to Y$, and $h_i : Y_i \to Y$, where $Y := \tilde{X}/K$, $Z := \tilde{X}/(H \cap K)$, H is a point stabilizer of G, $K := G \cap S_\Delta^I$ and K_i is a point stabilizer in the action of K on the i-th coordinate of Δ^I. Put $t := |I|$, $\ell := |\Delta|$, and $m := [G : K]$. Recall that by Remark 2.15 the fiber product of h_i and h_j is irreducible for distinct i, j in $\{1, \ldots, t\}$.



Theorem 7.1. *Fix $t \geq 2$ and $\alpha > 0$. Then there exist constants $c_2 = c_{2,t} > 0$ and $d_2 = d_{2,\alpha,t}$, depending only on α and t, such that for every indecomposable covering $f : X \to \mathbb{P}^1$ with monodromy group $G \leq S_\ell \wr S_t$ of product type, whose corresponding coverings h_i, $i = 1, \ldots, t$ admit a pairwise genus bound $\alpha\ell$, one of the following holds:*

(1) $g_X > c_2\ell^{t-1} - d_2\ell^{t-2}$,
(2) *$t = 2$ and the ramification of f appears in Table 3.1.*

Note that the proof shows that one can choose $c_2 = 1/(3m)$ for all $t \geq 3$. Note that m is bounded by $m \leq t!$, a constant depending only on t. We first deduce Theorem 1.1 from Theorem 7.1:

Proof of Theorem 1.1. The reader who is not interested in explicit constants is encouraged to skip the first paragraph and replace all terms of the form $E\ell^{t-2}$ where E depends on α and t by an expression of the form $O_{\alpha,t}(\ell^{t-2})$. Let c_2, d_2 be the constants from Theorem 7.1 with respect to a constant α, depending only on t, such that $\alpha \geq 2m + 1/2$. Note that we may choose such α, since m is bounded by $t!$, which depends only on t. Let E_0, and $L_{2,\alpha}$ be the constants from Proposition 5.1 and Corollary 4.4, respectively. Pick $c \leq \min\{1/2, 5/m, c_2\}$ and let $d > \max\{d_2, 5E_0/m\}$ be sufficiently large so that $d/c \geq \max\{6, 4E_0, L_{2,\alpha}\}$. Since the assertion is trivial if $c\ell - d < 0$, we may assume that $\ell \geq d/c \geq \max\{6, 4E_0, L_{2,\alpha}\}$. Since $c\ell^{t-1} - d\ell^{t-2} < \ell^{t-1}$, it suffices to prove the claim when $g_X < \ell^{t-1}$.

Let $\pi : Z \to Y$, $\pi_0 : Y \to X_0$ be the natural projections. We first prove the theorem under the assumption that the set S of branch points of π_0 is of cardinality $|S| \geq 5$.

Consider the covering $\tilde{f} := f \circ \pi = \pi_0 \circ h$. Since π_0 is Galois and $e_{\pi_0}(P) \geq 2$ for $P \in S$, we have $e_{\tilde{f}}(Q) \geq 2$ for every $Q \in \tilde{f}^{-1}(S)$. Thus,

$$R_{\tilde{f}}(P) \geq \frac{\deg \tilde{f}}{2} = \frac{m}{2}\ell^t \tag{7.1}$$

for every $P \in S$. By Proposition 5.1.(1), we also have

$$R_\pi(f^{-1}(P)) \leq m\ell^{t-1}/2 + E_0\ell^{t-2} \tag{7.2}$$

for every $P \in S$. Since $R_{\tilde{f}}(P) = mR_f(P) + R_\pi(f^{-1}(P))$ by the chain rule (2.1), we deduce from (7.1) and (7.2) that

$$R_f(P) = \frac{R_{\tilde{f}}(P) - R_\pi(f^{-1}(P))}{m} \geq \frac{\ell^t - \ell^{t-1}}{2} - \frac{E_0}{m}\ell^{t-2}.$$

As $g_{X_0} \geq 0$, together with the Riemann–Hurwitz formula for f this gives

$$\begin{aligned} 2(g_X - 1) &\geq 2\ell^t(g_{X_0} - 1) + \sum_{P \in S} R_f(P) \geq -2\ell^t + |S|\Big(\frac{\ell^t - \ell^{t-1}}{2} - \frac{E_0}{m}\ell^{t-2}\Big) \\ &= (|S| - 4)\Big(\frac{\ell^t - \ell^{t-1}}{2} - \frac{E_0}{m}\ell^{t-2}\Big) - 2\ell^{t-1} - \frac{4E_0}{m}\ell^{t-2}. \end{aligned} \tag{7.3}$$

As $\ell \geq \max\{6, E_0\}$, one has $(\ell^t - \ell^{t-1})/2 - E_0\ell^{t-2}/m > 0$. As in addition $\ell \geq 6$, and $c_2 \leq 1/2$, and $d_2 > 5E_0/m$, (7.3) in the case $|S| \geq 5$ gives

$$2(g_X - 1) \geq \frac{\ell^t - 5\ell^{t-1}}{2} - \frac{5E_0}{m}\ell^{t-2} \geq \frac{\ell^{t-1}}{2} - \frac{5E_0}{m}\ell^{t-2} > c\ell^{t-1} - d\ell^{t-2}.$$

Henceforth, assume $|S| \leq 4$. We next prove the theorem in the case $g_Y > 1$. By the Riemann–Hurwitz formula for h and π, one has

$$2m(g_X - 1) + R_\pi = 2(g_Z - 1) \geq 2\ell^t(g_Y - 1), \tag{7.4}$$

where $R_\pi := \sum_{P \in X_0(\mathbb{K})} R_\pi(f^{-1}(P))$. Since $|S| \leq 4$, Proposition 5.1.(1) implies

$$R_\pi \leq |S|\Big(\frac{m}{2}\ell^{t-1} - E_0\ell^{t-2}\Big) \leq 2\ell^{t-1} - 4E_0\ell^{t-2}.$$

Hence for $g_Y > 1$, as $\ell \geq 6$, and $c \leq 5/m$, and $d > 4E_0/m$, (7.4) gives

$$g_X > \frac{\ell^t - \ell^{t-1}}{m} - \frac{4E_0}{m}\ell^{t-2} > \frac{5}{m}\ell^{t-1} - \frac{4E_0}{m}\ell^{t-2} > c\ell^{t-1} - d\ell^{t-2}.$$

Henceforth assume that $g_Y \leq 1$. Note that this implies $g_{X_0} = 0$ as follows. Clearly $g_{X_0} \leq g_Y \leq 1$. If $g_{X_0} = 1$ then π_0 is unramified. In the latter case, all branch cycles of f are contained in S_Δ^I and hence $G \leq S_\Delta^I$, which by Lemma 2.9 contradicts the transitive action of G on I.

Since $g_Y \leq 1$, Proposition 6.1 gives $R_\pi \leq \max\{2m\ell^{t-1} + 4E_0\ell^{t-2}, 2tE_0\ell^{t-2}\}$. As in addition $g_X < \ell^{t-1}$, and $\ell \geq 4E_0$, and $m \geq t$, this bound on R_π and the Riemann-Hurwitz formula for π give

$$2(g_Z-1)=2m(g_X-1)+R_\pi<\max\{4m\ell^{t-1}+4E_0\ell^{t-2},2m\ell^{t-1}+2tE_0\ell^{t-2}\}$$
$$\leq(4m+1)\ell^{t-1}\leq 2\alpha\ell^{t-1}.$$

Thus, letting $Y_{i,j}:=\tilde{X}/(K_i\cap K_j)$ for distinct $i,j\in I$, the Riemann–Hurwitz formula for the natural projection $Z\to Y_{i,j}$ gives $g_{Y_{i,j}}-1\leq\frac{1}{\ell^{t-2}}(g_Z-1)<\alpha\ell$. The natural projection $Y_{i,j}\to Y$ is a minimal covering which factors through h_i and h_j, and its degree is $\deg h_i\cdot\deg h_j=\ell^2$, for distinct $i,j\in I$. By assumption we also have $\ell\geq L_{2,\alpha}$. It follows that the coverings h_i, $i\in I$ admit a pairwise genus bound $\alpha\ell$. We can therefore apply Theorem 7.1, and deduce that either $g_X>c_2\ell^{t-1}-d_2\ell^{t-2}$, or $t=2$ and the ramification of f appears in Table 3.1. In the latter case, G contains A_ℓ^2 by Lemma 3.3, and $g_X\leq 1$ by Remark 3.3. □

We next show that Theorem 7.1 for $t\geq 3$ follows from:

Proposition 7.2. *Let $t\geq 3$ and $\alpha>0$. There exists a constant $E_2=E_{2,t,\alpha}>0$, depending only on α and t, which satisfies the following property. For every covering $f:X\to\mathbb{P}^1$ with monodromy group $G\leq S_\ell\wr S_t$ of product type, such that its associated coverings h_i, $i=1,\dots,t$ admit a pairwise genus bound $\alpha\ell$, the Riemann–Hurwitz contribution of the associated covering π is bounded by $\sum_{P\in\mathbb{P}^1(\mathbb{K})}R_\pi(f^{-1}(P))<E_2\ell^{t-2}$.*

We note that Proposition 7.2 follows from Proposition 6.1 in case none of the branch cycles acts on I as a transposition. Proposition 7.2 is proved in Section 8.

Proof of Theorem 7.1 for $t\geq 3$. We claim that $g_Z-1>\ell^{t-1}/6-d_2'\ell^{t-2}$ for a constant d_2' depending only on t and α. Since the contribution $R_\pi:=\sum_{P\in X(\mathbb{K})}R_\pi(f^{-1}(P))$ is bounded by $E_2\ell^{t-2}$ by Proposition 7.2, the theorem follows from the claim by applying the Riemann–Hurwitz formula for π:

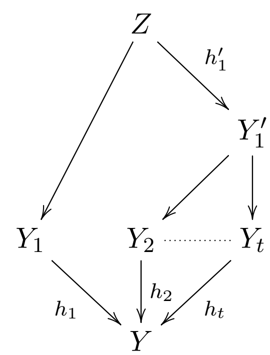
$$2(g_X-1)=\frac{1}{m}(2(g_Z-1)-R_\pi)>\frac{1}{3m}\ell^{t-1}-\frac{2d_2'+E_2}{m}\ell^{t-2},$$

by choosing $c_2\leq 1/(3m)$ and $d_2\geq(2d_2'+E_2)/m$, depending only on t. Note this is possible since as usual m is bounded by a constant $t!$ depending only on t.

To prove the claim, we apply the Riemann–Hurwitz formula to the natural projection $h_1':Z\to Y_1'$, where $Y_1':=\tilde{X}/(K_2\cap\cdots\cap K_t)$. Recall that as G is of product type, Remark 2.12.(1) implies $[K:\bigcap_{j=2}^t K_j]=\ell^{t-1}$ and hence that $\deg h_1'=[\bigcap_{j=2}^t K_j:\bigcap_{j=1}^t K_j]=\ell$.

Note that as G is of product type and hence nonsolvable, and since h_i, $i=1,\dots,t$ admit a pairwise genus bound $\alpha\ell$, Lemma 4.7 implies that there exists $P_0\in Y(\mathbb{K})$ of type

$m_0 = m_{h_1}(P_0) < \infty$, and $r \in E_{h_1}(P_0)$ that does not divide m_0.



Let U be the set of preimage $Q_Z \in h^{-1}(P_0)$ whose image Q_i in Y_i has ramification index $e_{h_i}(Q_i) = r$ if $i = 1$ and $e_{h_i}(Q_i) = m_0$ if $i = 2, \dots, t$. Since h is a minimal covering which factors through h_i, $i \in I$, and $\deg h = \deg h_1 \cdots \deg h_t$ by Remark 2.15.(1), Abyahnkar's Lemma 2.2 implies that for each choice of preimages $Q_i \in h_i^{-1}(P_0)$, with $e_{h_i}(Q_i) = r$ if $i = 1$ and $e_{h_i}(Q_i) = m_0$ if $i = 2, \dots, t$, there exist $(r, m_0) \cdot m_0^{t-2}$ points of Z with image Q_i in Y_i, $i = 1, \dots, t$. Let $\varepsilon = \varepsilon_{\alpha, m_0}$ be the error from Definition 4.2. Since $m_0 = m_{h_i}(P_0) < \infty$ for all $i \in I$, there are at least $(\ell - \varepsilon)/m_0$ preimages $Q_i \in h_i^{-1}(P_0)$ such that $e_{h_i}(Q_i) = m_0$, for each $i = 2, \dots, t$. Thus in total

$$|U| \geq (r, m_0) \cdot m_0^{t-2} \left(\frac{\ell - \varepsilon}{m_0}\right)^{t-1} \geq \frac{(r, m_0)}{m_0} \ell^{t-1} - E_3 \ell^{t-2}, \tag{7.5}$$

for some constant E_3 depending only on t and α. Moreover, Abhyankar's Lemma 2.2 implies $e_{h_1'}(Q_Z) = r/(r, m_0)$ for all $Q_Z \in U$. Hence the Riemann-Hurwitz formula for $h_1' : Z \to Y_1'$ and (7.5) give:

$$\begin{aligned} 2(g_Z - 1) &\geq 2\ell(g_{Y_1'} - 1) + \sum_{Q_Z \in U} (e_{h_1'}(Q_Z) - 1) \geq -2\ell + |U|\big(\frac{r}{(r, m_0)} - 1\big) \\ &> -2\ell + \frac{(r, m_0)}{m_0}\big(\frac{r}{(r, m_0)} - 1\big)\ell^{t-1} - E_3\big(\frac{r}{(r, m_0)} - 1\big)\ell^{t-2} \\ &> \frac{r - (r, m_0)}{m_0}\ell^{t-1} - 2d_2'\ell^{t-2}, \end{aligned} \tag{7.6}$$

for some constant d_2', depending only on t, such that $2d_2' > 2/\ell^{t-3} + (r - (r, m_0))E_3/(r, m_0)$. Note that such a d_2' exists since r is bounded by the error ε which depends only on α and $m_0 \leq 6$. As $m_0 \leq 6$, as $m_0 \neq 5$, and as r does not divide m_0, we have $(r - (r, m_0))/m_0 \geq 1/3$. Thus (7.6) gives $g_Z - 1 > \ell^{t-1}/6 - d_2'\ell^{t-1}$, proving the claim, and hence the theorem. □

The proof of Theorem 7.1 for $t = 2$ is given in Section 9.2.

8. Reduced tuples: a proof of Theorem 7.1 for $t \geq 3$

In this section we complete the proof of Proposition 7.2, and hence that of Theorem 7.1 for $t \geq 3$. We assume the setup of §2.7 with $X_0 := \mathbb{P}^1$, so that $f : X \to \mathbb{P}^1$ is a covering with primitive monodromy group $G \leq S_\Delta \wr S_I$ of product type with $t = |I| \geq 3$, Galois closure $\tilde{X}$, and $K := G \cap S_\Delta^I$. As outlined in Section 1, we shall create from f a new covering $\hat{f}$ whose monodromy is a subgroup of S_Δ. This is done in the following proposition which creates out of a product 1 tuple for f, a product 1 tuple in S_Δ consisting of reduced forms of branch cycles of f.

Definition 8.1. Let $x_1, \dots, x_r \in G$ be a product 1 tuple for a group $G \leq S_\Delta \wr S_I$ of product type, and let O_j be the set of orbits of x_j on I. Put $J := \{1, \dots, r\}$. A multiset $T := \{y_{\theta,j} \in S_\Delta : \theta \in O_j, j \in J\}$, is called a *reduced product* 1 *multiset* of $x_1, \dots, x_r$ if it satisfies the following properties:

1. there exists a reduced form $y_j = a_j\sigma_j$, $a_j \in S_\Delta^I$, $\sigma_j \in S_I$ of x_j with representatives ι_θ, $\theta \in O_j$ such that $a_j(\iota_\theta) = y_{\theta,j}$, for all $\theta \in O_j$, $j \in J$;
2. the group $\langle T \rangle$ acts transitively on Δ;
3. for some choice of $\omega_{\theta,j} \in \{1, -1\}$, $\theta \in O_j$, $j \in J$ and in some ordering, the elements $y_{\theta,j}^{\omega_{\theta,j}}$, $\theta \in O_j$, $j \in J$ have product 1.

Let $\overline{\pi}_0 : \overline{Y} \to \mathbb{P}^1$ be the natural projection from the quotient $\overline{Y}$ by a point stabilizer in the action of G on I. Since G is of product type, it acts transitively on I by Lemma 2.9, and hence this action is equivalent to the G-action on $I\backslash G$. As K is the kernel of this action, the natrual projection $\pi_0 : Y \to \mathbb{P}^1$ from $Y := \tilde{X}/K$, is the Galois closure of $\overline{\pi}_0$, and $\overline{G} := \mathrm{Mon}(\overline{\pi}_0)$ is isomorphic to G/K equipped with its action on I. Put $t := \deg \overline{\pi}_0 = |I|$, and $m := \deg \pi_0 = [G : K]$.

Proposition 8.2. *Assume $g_{\overline{Y}} = 0$, $g_Y \leq 1$, and let $x_1, \dots, x_r$ be a product* 1 *tuple for G whose images $\sigma_1, \dots, \sigma_r \in \overline{G}$ satisfy*[3]*:*

(LG_Y) *$\sigma_i = 1$ if and only if $i > s$ for some $s \in \{3, 4\}$, and $\langle \sigma_1, \sigma_2 \rangle = \langle \sigma_3, \sigma_4 \rangle = \overline{G}$ if $s = 4$, and σ_1 is a t-cycle if $s = 3$.*

Then there exists a reduced product 1 *multiset for $x_1, \dots, x_r$.*

Since the proof of Proposition 8.2 is involved, for the benefit of the reader we give the following example which illustrates our process of generating a product 1 multiset.

Example 8.3. Suppose $t = 3$, and x_1, x_2, x_3 is a product 1 tuple for G whose images in $G/K \leq S_3$ are r, s, rs, respectively, where $r = (1, 2, 3)$ and $s = (1, 2)$. In this example, we form a reduced product 1 multiset for G. We next claim that

1. x_1, x_2, x_3 have reduced forms $y_1 = (\alpha, 1, 1)r$, $y_2 = (1, b_1, b_2)s$, and $y_3 = (c_1, c_2, 1)rs$, respectively, where $r = (1, 2, 3)$, $s = (1, 2)$, and $a, b_1, b_2, c_1, c_2 \in S_\ell x$;
2. $W := \{a, b_1, b_2, c_1, c_2\}$ is a reduced product 1 multiset for x_1, x_2, x_3.

[3]The condition (LG_Y) is required in the proposition, only for the sake of simplicity of its proof. We do not know of counterexamples to the assertion when this condition is not assumed.

Proof of claim. Since the reduced form of x_1 is $y_1 := (a,1,1)r$ for some $a \in S_\Delta$, there exists $z \in S_\Delta^I$ such that $x_1^z = y_1$, by Lemma 2.6. For simplicity replace the tuple x_1, x_2, x_3 by the tuple x_1^z, x_2^z, x_3^z, and G by G^z. Then $x_1 = y_1$ is in reduced form. Write $x_2 = (\beta_1, \beta_2, \beta_3)s$ and $x_3 = (\gamma_1, \gamma_2, \gamma_3)rs$. Since G is transitive on Δ^I, the subgroup $K_0 = \langle a, \beta_i, \gamma_i, i = 1,2,3\rangle \le S_\Delta$ is transitive. Note that the orbits of s are $\mu = \{1,2\}$ and $\eta = \{3\}$ and hence x_2 has a reduced form $y_2 = (1, b_1, b_2)s$ where $b_1 = \beta_2\beta_1$ and $b_2 = \beta_3$, by Lemma 2.6. Similarly, x_3 has a reduced form $y_3 = (c_1, c_2, 1)rs$ where $c_2 = \gamma_2\gamma_3$ and $c_1 = \gamma_1$.

The product 1 relation gives:

$$\begin{aligned} 1 = x_1x_2x_3 &= (a,1,1)r \cdot (\beta_1,\beta_2,\beta_3)s \cdot (\gamma_1,\gamma_2,\gamma_3)rs \\ &= (a,1,1) \cdot (\beta_1,\beta_2,\beta_3)^{r^{-1}} \cdot (\gamma_1,\gamma_2,\gamma_3)^{rs}. \end{aligned}$$

Since $(\beta_1,\beta_2,\beta_3)^{\sigma^{-1}} = (\beta_{1^\sigma}, \beta_{2^\sigma}, \beta_{3^\sigma})$ for every $\sigma \in S_t$, the product 1 relation amounts to the equalities:

$$\begin{aligned} a\beta_2\gamma_1 &= 1 \\ \beta_3\gamma_3 &= 1 \text{ or equivalently } \gamma_3\beta_3 = 1 \\ \beta_1\gamma_2 &= 1. \end{aligned}$$

By iteratively inserting the equalities into the first one, we get:

$$1 = a\beta_2\gamma_1 = a\beta_2\beta_1\gamma_2\gamma_1 = a \cdot (\beta_2\beta_1) \cdot (\gamma_2\gamma_3) \cdot \beta_3 \cdot \gamma_1 = ab_1c_2b_2c_1.$$

It remains to show that W generates K_0 since then $\langle W\rangle$ is transitive. This holds since by the above equalities $\gamma_3 = \beta_3^{-1} \in \langle\beta_3\rangle$ and $\beta_1 = \gamma_2^{-1} \in \langle\gamma_2\rangle$ and hence

$$\langle W\rangle = \langle a, \beta_2\beta_1, \beta_3, \gamma_1, \gamma_2\gamma_3\rangle = \langle a, \beta_2\beta_1, \beta_3, \gamma_1, \gamma_2\rangle = \langle a, \beta_2, \beta_3, \gamma_1, \gamma_2\rangle = K_0.$$

This completes the proof of the claim. □

Proposition 8.2 generalizes this argument under the mere condition (LG_Y).

We first note that if one of the x_i's acts on I as a transposition, the product 1 tuple can be modified so that condition (LG_Y) holds

Lemma 8.4. *Assume that* $\deg \overline{\pi}_0 > 2$ *and that the ramification type of* $\overline{\pi}_0$ *appears in Table 6.1. Then there exists a product* 1 *tuple* $x_1, \dots, x_r$ *for* G *whose images in* $\overline{G}$ *satisfy (*LG_Y*), and whose ramification type coincides with the ramification type of* f*.*

Proof. Let $\mathcal{R}$ be the ramification type of f. As in Section 2.4, there exists a product 1 tuple $x_1, \dots, x_r$ for G with ramification type $\mathcal{R}$. Write $x_j = a_j\sigma_j$, $a_j \in S_\Delta^I$, $\sigma_j \in S_I$, $j = 1, \dots, r$. Note that by swapping $(x_i, x_{i+1}) \to (x_{i+1}, x_i^{x_{i+1}})$ for some $i \in \{1, \dots, r\}$, one obtains another product 1 tuple for G with ramification type $\mathcal{R}$. By making such swaps, we may assume $x_1, \dots, x_r$ is a product 1 tuple for G with ramification type $\mathcal{R}$, whose images $\sigma_1, \dots, \sigma_r \in \overline{G}$ are ordered by decreasing element order. In particular $\sigma_i = 1$ if and only if $i > s$ for some s.

The ramification type corresponding to $\sigma_1, \dots, \sigma_r$ appears in Table 6.1. Hence $s \in \{3,4\}$. Moreover, every ramification type with $s = 3$ contains the multiset $[t]$, and among the multisets in such a ramification type, $[t]$ has the largest least common multiple. Hence, if the tuple $x_1,\dots,x_r$ corresponds to such a ramification type, then the element σ_1 of highest order among the σ_i's is a t-cycle, so that (LG$_Y$) holds.

Henceforth assume $s=4$. There are two ramification types in Table 6.1 with $s=4$ and $\deg\overline{\pi}_0>2$, namely, $[2,1]$ four times with $t=3$, and $[2,1^2]$ twice, $[2,2]$ twice with $t=4$. In both cases, $\sigma_1,\dots,\sigma_4$ are involutions, and $\overline{G}\cong D_{2t}$. Note that in both cases a direct inspection shows that every product 1 tuple for D_{2t}, $t\in\{3,4\}$, consisting of four involutions does not contain a rotation, and hence consists of reflections.

Assume first that $t=3$. As $\overline{G}=\langle\sigma_1,\dots,\sigma_4\rangle$, by possibly swapping $(x_2,x_3)\to(x_3,x_2^{x_3})$, we may assume that $\sigma_1\neq\sigma_2$ and hence that $\sigma_3\neq\sigma_4$ by the product 1 relation. Since every two distinct reflections generate D_6, we have $\langle\sigma_1,\sigma_2\rangle=\langle\sigma_3,\sigma_4\rangle=D_6$, so that (LG$_Y$) holds.

Now assume $t=4$. There are two types of reflections in D_8, transpositions and products of two transpositions. Moreover, two reflections generate D_8 if and only if they are of distinct types. The ramification type in Table 6.1 with $s=4$ and $t=4$ consists of two elements of each type. By possibly swapping $(x_2,x_3)\to(x_3,x_2^{x_3})$, we may assume σ_1 and σ_2 are of distinct type, and hence so are σ_3 and σ_4, giving $\langle\sigma_1,\sigma_2\rangle=\langle\sigma_3,\sigma_4\rangle=D_8$, so that (LG$_Y$) holds. ☐

The proof of Proposition 8.2 relies on the following lemma:

Lemma 8.5. *Assume that (LG$_Y$) holds with $s\in\{3,4\}$, and that $g_{\overline{Y}}=0$ and $g_Y\le 1$. Then $|\mathrm{Orb}_I(\sigma_{s-1})|+|\mathrm{Orb}_I(\sigma_s)|=t+1$. Moreover, $|\mathrm{Orb}_I(\sigma_1)|+|\mathrm{Orb}_I(\sigma_2)|=t+1$ if $s=4$.*

Proof. For $s=3$, this follows directly as σ_1 has one orbit and the Riemann–Hurwitz formula for $\overline{\pi}_0$ gives $\sum_{i=1}^3|\mathrm{Orb}_I(\sigma_i)|=t+2$. Assume $s=4$. Since $g_Y\le 1$ and $\pi_0:Y\to\mathbb{P}^1$ is Galois with four branch points, the Riemann–Hurwitz formula for π_0 implies that the elements $\sigma_1,\dots,\sigma_4$ are involutions. Moreover, as (LG$_Y$) holds, the group $\overline{G}:=\mathrm{Mon}(\overline{\pi}_0)$ is generated by two involutions and hence $\overline{G}\cong D_m$ is a Dihedral group. Since $\overline{\pi}_0$ is not Galois as $g_{\overline{Y}}<g_Y$, the action of $\overline{G}$ is faithful, transitive and nonregular, and hence its the natural action of $\overline{G}\cong D_m$ on a regular t-gon for $t=m/2$. Moreover, since there is no product 1 four tuple of involutions generating $\overline{G}$ which contains a rotation, the elements σ_i, $i=1,\dots,4$ are reflections. If t is odd, the number of orbits of every reflection is $(t+1)/2$ and the claim follows. If t is even, there are two types of reflections, those with no fixed point and those with two fixed points, which have $t/2$ or $t/2+1$ orbits, respectively. Since two reflections that generate D_{2t} are necessarily of distinct types, the assumption $\langle\sigma_3,\sigma_4\rangle=D_{2t}$ implies that $\{|\mathrm{Orb}_I(\sigma_3)|,|\mathrm{Orb}_I(\sigma_4)|\}=\{t/2,t/2+1\}$. Since $g_{\overline{Y}}=0$, the Riemann–Hurwitz formula for $\overline{\pi}_0$ then forces $\{|\mathrm{Orb}_I(\sigma_1)|,|\mathrm{Orb}_I(\sigma_2)|\}=\{t/2,t/2+1\}$. ☐

Proof of Proposition 8.2. For $1\le j\le r$, write $x_j=a_j\sigma_j$ where $a_j\in S_\Delta^I$, $\sigma_j\in S_I$, so that $\overline{G}\cong G/K$ is generated by $\sigma_1,\dots,\sigma_r$.

Step I: *We first define an element $z\in S_\Delta^I$ such that x_1^z,x_2^z (resp., x_1^z) are in reduced form if $s=4$ (resp., $s=3$).* We define z on I iteratively. First, assume z is defined on a subset $I_1\subseteq I$, and define it on a larger subset of I. Put $J_1:=\{1,\dots,s-2\}$, $J_2:=\{s-1,s\}$, and $J_3:=\{s+1,\dots,r\}$. Initially, set I_1 to be a subset of I consisting of a single element denoted by $1 \in I$, set $z(1) = 1$, and set $P = \emptyset$. Since $\langle \sigma_j : j \in J_1 \rangle$ acts transitively on I, there exists some $i \in I \setminus I_1$ and $j \in J_1$ such that $i^{\sigma_j} \in I_1$ if $s = 4$ (resp., if $s = 3$). Then set

$$z(i) := a_j(i) z(i^{\sigma_j}).$$

Adding i to I_1 and the pair $(j,i) \in P$, we can repeat this process until $I_1 = I$.

Writing $x_j^z = b_j\sigma_j$ for $b_j \in S_\Delta^I$, we have $b_j = z^{-1}a_j z^{\sigma_j^{-1}}$ and hence

$$b_j(i) = (z^{-1}a_j z^{\sigma_j^{-1}})(i) = z^{-1}(i)a_j(i)z(i^{\sigma_j}) = 1, \text{ for all } (j,i) \in P.$$

Moreover, it follows that for every $i \in I \setminus \{1\}$, there exists $j \in J_1$ such that $(i,j) \in P$. For $s = 3$, this means that P contains all pairs $(1,i)$, $i \in I \setminus \{1\}$, so that $b_1(i) = 1$, for $i \in I \setminus \{1\}$, and hence x_1^z is in reduced form. For $s = 4$, one has $(t - |\mathrm{Orb}_I(\sigma_1)|) + (t - |\mathrm{Orb}_I(\sigma_2)|) = t - 1 = |P|$ by Lemma 8.5. Since by construction the set $P_j := \{i \,|\, (j,i) \in P\}$ does not contain any orbit of σ_j for $j \in J_1$. It follows that every orbit θ of σ_j contains exactly one $\iota_\theta \in \theta$ such that $(j,\iota_\theta) \notin P$, so that $b_j(i) = 1$ for all $i \in \theta \setminus \{i_\theta\}$. Hence, x_j^z is in reduced form for $j = 1,2$.

Since a reduced form of x_j^z is also a reduced form of x_j, we may replace the elements x_j by x_j^z, for all $j = 1,\ldots,r$. Thus we may assume that x_j, $j \in J_1$ are in reduced form. We note that since $x_1,\ldots,x_r$ act transitively on Δ^I, the subgroup $K_0 \le S_\Delta$ generated by the elements $a_j(i)$, $j = 1,\ldots,r$, $i \in I$, acts transitively on Δ. Since $a_j(i) = 1$ for $(j,i) \in P$ we shall henceforth consider the following multiset of generators for K_0:

$$W := \{a_j(i) \,|\, j = 1,\ldots,r, i \in I, \text{ and } (j,i) \notin P\}.$$

Step II: *Describing the product* 1 *relation.* Consider the product 1 relation:

$$1 = \prod_{j=1}^r x_j = \prod_{j=1}^r a_j\sigma_j = \prod_{j=1}^r \sigma_j \cdot \prod_{j=1}^r a_j^{\tau_j},$$

where $\tau_j = \prod_{k=j}^r \sigma_k$, $j = 1,\ldots,r$. Since $\tau_1 = \prod_{j=1}^r \sigma_j = 1$, we have $\prod_{j=1}^r a_j^{\tau_j} = 1$. Evaluating the latter expression at $i \in I$, we get:

$$\prod_{j=1}^r a_j^{\tau_j}(i) = \prod_{j=1}^r a_j(i^{\tau_j^{-1}}) = 1,\ i \in I. \tag{8.1}$$

Step III: *Describing an iterative procedure to form the product* 1 *relation from the* t *equalities in* (8.1) Let O_j be the set of orbits of σ_j on I. By Lemma 2.6, there exist reduced forms $y_j = c_j\sigma_j$, $c_j \in S_\Delta^I$, $\sigma_j \in S_I$, of x_j such that

$$c_j(\iota_\theta) = a(\iota_\theta)\cdots a(\iota_\theta^{\sigma_j^{|\theta|-1}}),$$

for $\theta \in O_j$, $j \in J_2$. We put an ordering on each orbit $\theta \in O_j$ by setting $i_1 < i_2$ for $i_1, i_2 \in \theta$ if $u_1 < u_2$, where u_k is the minimal nonnegative integer such that i_k is the image of ι_θ under $\sigma_j^{u_k}$ for $k = 1,2$.

At each step of the process we shall modify (8.1) with $i = 1$, so that it will consist of a product over all elements $a_j^{\tau_j}(i)$, each appearing once with exponent 1 or -1, where i runs through a proper subset $I_2 \subseteq I$, and $j \in J_1 \cup J_2 \cup J_3$, with the exception of pairs (j,i) in a set Q associated with this process.

(Step A) Initially, set I_2 to be the subset of I consisting of the element 1, and set $Q = \emptyset$.

Since I_2 is a proper subset of I, and $\langle \sigma_{s-1}, \sigma_s \rangle = \langle \sigma_{s-1}^{\tau_{s-1}}, \sigma_s \rangle = \overline{G}$ is transitive on I, there exist $\mu \in I_2$ and $k \in J_2$ such that the orbit $\hat{\theta}$ of μ under $\sigma_k^{\tau_k}$ is not contained in I_2. For each $\nu \in \hat{\theta} \setminus I_2$, equality (8.1) gives

$$a_k(\nu^{\tau_k^{-1}}) = F_k, \text{ where } F_k := \prod_{j=k-1}^{1} a_j(\nu^{\tau_j^{-1}})^{-1} \cdot \prod_{j=s}^{k+1} a_j(\nu^{\tau_j^{-1}})^{-1}$$

is a product of the terms $a_j^{\tau_j}(\nu)^{-1}$ over all $j \neq k$, each appearing once. Note that since $(i^{\sigma_k^{\tau_k}})^{\tau_k^{-1}} = (i^{\tau_k^{-1}})^{\sigma_k}$ for $i \in I$, the set $\hat{\theta}^{\tau_k^{-1}} := \{i^{\tau_k^{-1}} : i \in \hat{\theta}\}$ is an orbit of σ_k. Next,

(Step B) Iteratively insert the expressions $F_k^{-1} \cdot a_k(\nu^{\tau_k^{-1}})$ (resp., $a_k(\nu^{\tau_k^{-1}})F_k^{-1}$), which equal 1, into equality (8.1) with $i = 1$ at the left (resp., right) of $a_k(\mu^{\tau_k^{-1}})$, so that the new equality at $i = 1$ contains a product $B_{k,\hat{\theta}} := \prod_{i \in (\hat{\theta} \setminus I_2) \cup \{\mu\}} a_k(i^{\tau_k^{-1}})$, ordered so that the indices $i^{\tau_k^{-1}}$ are increasing with respect to the ordering of the orbit $\hat{\theta}^{\tau_k^{-1}}$ of σ_k.

(Step C) Add $\hat{\theta} \setminus I_2$ to I_2; add the pair (k,ν) to Q for every $\nu \in \hat{\theta} \setminus I_2$; and replace in W the elements $a_k(i^{\tau_k^{-1}})$, $i \in \{\mu\} \cup (\hat{\theta} \setminus I_2)$, by the element $B_{k,\hat{\theta}}$.

We repeat steps (B)-(C) until $I_2 = I$, in which case the process terminates.

Note that after preforming step (B) with an orbit $\hat{\theta}$ of $\sigma_k^{\tau_k}$, equality (8.1) contains all elements $a_k^{\tau_k}(i)$, $i \in \hat{\theta}$, and therefore these elements and the product $B_{k,\hat{\theta}}$ remain unchanged in further steps of the process. For each pair (j,i), $j \in J_2$, $i \in I$, for which $a_j^{\tau_j}(i)$ appears at the resulting equality (8.1) but not as part of a product $B_{j,\hat{\theta}}$, denote $B_{j,\{i\}} := a_j^{\tau_j}(i)$, for $j \in J_2$. We shall see that in such cases $\{i\}$ is a length 1 orbit of $\sigma_j^{\tau_j}$.

Step IV: *We show that the multiset T consisting of the elements $a_j(\iota_\theta)$, $\theta \in O_j$, $j \in J_1 \cup J_3$, and $c_j(\iota_\theta)$, $\theta \in O_j$, $j \in J_2$ is a reduced product 1 multiset of $x_1, \ldots, x_r$, where the product 1 relation is given by the equality (8.1) resulting at the end of the process at $i = 1$.*

We claim that each element $c_j(\iota_\theta)$, $\theta \in O_j$, $j \in J_2$ appears in the resulting equality (8.1). Let $\hat{Q}$ denote the complement of Q in $\{(j,i) \colon j \in J_2, i \in I\}$. Since each $i \in I \setminus \{1\}$ is added to I_2 once, we have $|Q| = t-1$ and hence $|\hat{Q}| = t+1$. Thus, Lemma 8.5 implies that $|\mathrm{Orb}_I(\sigma_{s-1}^{\tau_{s-1}})| + |\mathrm{Orb}_I(\sigma_s^{\tau_s}))| = t+1 = |\hat{Q}|$. Since by construction, for every orbit $\hat{\theta}$ of $\sigma_j^{\tau_j}$, $j \in J_2$, there exists at least one $\mu \in \hat{\theta}$ such that $(j,\mu) \in \hat{Q}$, the latter equality and $|\hat{Q}| = t+1$ imply that every such orbit $\hat{\theta}$ contains exactly one $\mu \in \hat{\theta}$ such that $(j,\mu) \in \hat{Q}$. Thus, $B_{j,\hat{\theta}}$ is a product of elements $a_j^{\tau_j}(i)$ running over all $i \in \hat{\theta}$, for every orbit $\hat{\theta}$ of $\sigma_j^{\tau_j}$, $j \in J_2$. In particular, the elements $B_{j,\{i\}} = a_j^{\tau_j}(i)$ defined in the end of Step III are products over length 1 orbits $\{i\}$ of $\sigma_j^{\tau_j}$, $j \in J_2$, and at each step B we have $\hat{\theta} \cap I_2 = \{\mu\}$. Since $\theta := \hat{\theta}^{\tau_j^{-1}}$ is an orbit of σ_j, we have $B_{j,\hat{\theta}} = c_j(\iota_\theta)$ by definition of $B_{j,\hat{\theta}}$, for every orbit $\hat{\theta}$ of $\sigma_j^{\tau_j}$, $j \in J_2$. Since there is a one to one correspondence between orbits of σ_j and $\sigma_j^{\tau_j}$, this shows that each element $c_j(\iota_\theta)$, $\theta \in O_j$, $j \in J_2$ appears once in the resulting product (8.1) for $i = 1$, proving the claim. Since each of the equations (8.1) for $i \in I$ is inserted exactly once into the equation with $i = 1$, it follows that resulting equality (8.1) is a product 1 relation consisting of the elements in T, each appearing once either with exponent 1 or with exponent -1.

It remains to prove the transitivity of T. We claim that the resulting set W generates the same subgroup K_0 and hence is transitive. By definition of $B_{k,\hat{\theta}}$ and since we have shown that $\hat{\theta} \cap I_2 = \{\mu\}$ at each step B, the element $a_k^{\tau_k}(\mu)$ is in the group generated by $B_{k,\hat{\theta}}$ and the elements $a_k^{\tau_j}(i)$, $i \in \hat{\theta} \setminus \{\mu\}$, and hence by replacing it with $B_{k,\hat{\theta}}$, one still has $\langle W \rangle = K_0$. By equality (8.1), $a_k^{\tau_k}(i)$ is in the group generated by the elements $a_j^{\tau_j}(i)$, $j \neq k$, and hence after removing the elements $a_k^{\tau_k}(i)$, $i \in \hat{\theta} \setminus \{\mu\}$ from W in step C, one still has $\langle W \rangle = K_0$. In total after each step C, one has $\langle W \rangle = K_0$, proving the claim and hence completing the proof. □

To the covering $f : X \to \mathbb{P}^1$, one associates the natural projections $h_i : \tilde{X}/K_i \to \tilde{X}/K$ and $\pi_0 : \tilde{X}/K \to \mathbb{P}^1$, where K_i is a point stabilizer in the action of K on the i-th copy Δ, for $i \in I$. Recall that by Remark 2.15.(1), the fiber product of h_i and h_j is irreducible for every two distinct $i, j \in \{1, \dots, t\}$. The following proposition shows that if the coverings h_i, $i \in I$ admit a pairwise genus bound $\alpha\ell$ and there is branch cycle of f, a power of which acts on I as a transposition, then the branch cycles of f are conjugate to elements $y\sigma \in S_\Delta \wr S_I$, $y \in S_\Delta^I$, $\sigma \in S_I$ such that $y(i)$, $i \in I$ are of almost Galois type with bounded error in the sense of Definition 5.2.

Proposition 8.6. *For every constant $\alpha > 0$, there exist constants $\hat{\varepsilon} := \hat{\varepsilon}_{\alpha,t}$ and N_α satisfying the following property. Let $x_j \in S_\Delta \wr S_I$, $j \in J$ be a product 1 tuple for f, and $h_i : Y_i \to Y$, $i \in I$ the coverings corresponding to f. Let O_j be the set of orbits of x_j on I, let e_j the order of the image of x_j in S_I, and put $z_j := x_j^{e_j} \in S_\Delta^I$.*

Assume that there exists a reduced product 1 multiset $y_{\theta,j}$, $\theta \in O_j$, $j \in J$ for the typle $x_1, \dots, x_r$. Assume that h_i, $i \in I$ admit a pairwise genus bound $\alpha\ell$. Finally assume that O_k consists of odd length orbits and a single length 2 orbit for some $k \in J$. Then $y_{\theta,j}$ and $z_j(\iota_\theta)$ are of almost Galois type with error at most $\hat{\varepsilon}$ for finite types, and with entry bound N_α for infinite types, such that $m(y_{\theta,j}) = m(z_j(\iota_\theta))$ if x_j does not act on I as a transposition or if $|\theta| > 1$, and $m(y_{\theta,j}) = 2m(z_j(\iota_\theta))$ otherwise, for $\theta \in O_j$, $j \in J$.

In fact, we show that $N = N_\alpha$ is simply the entry bound corresponding to the constant α in Definition 4.6. Recall that $m = \deg \pi_0$, and let $\varepsilon_j = \varepsilon_{j,\alpha}$ (resp. $\varepsilon = \varepsilon_\alpha$) denote the error (resp. total error) of h_i, $i \in I$ over the branch point of x_j. (Note that the error is independent of $i \in I$.) We show that $z_j(\iota_\theta)$ is of almost Galois type m_j with error at most ε_j, for all $\theta \in O_j$, $j \in J$. We also show the same error occurs for $y_{\theta,j}$ if x_j does not act on I as a transposition, or if $|\theta| > 1$. Otherwise, we show that $y_{\theta,j}$ is almost Galois with error at most $6m_j m(\varepsilon + N)$, for all $\theta \in O_j$, $j \in J$.

Since the proof is subtle as it considers many cases, we first demonstrate its outline in the following example:

Example 8.7. Following Example 8.3, let x_1, x_2, x_3 be a product 1 tuple for a covering f with monodromy group $G = S_\ell \wr S_3$ with reduced forms $y_1 = (a, 1, 1)r$, $y_2 = (b_1, 1, b_2)s$, and $y_3 = (c_1, c_2, 1)rs$, respectively, such that $W := \{a, b_1, b_2, c_1, c_2\}$ is a reduced product 1 multiset.

Assume that the associated coverings $h_i : Y_i \to Y$, $i \in I$ admit a pairwise genus bound $\alpha\ell$, and the list of values $m_{h_1}(Q)$, $Q \in Y$ that are greater than 1, is $\{3, 3, 3\}$. We then claim that a, b_1, b_2, c_1, c_2 are permutations of almost Galois type with error at most 30ε, where $\varepsilon := \varepsilon_{\alpha,3}$ is the error for points of type 3. Here, a permutation is of "almost Galois type" in the sense of Definition 5.2. We note that in this example the resuling Galois types in W that are greater than 1 are $\{2, 3, 6\}$ and not $\{3, 3, 3\}$.

Proof of claim (from Example 8.7). We first describe the ramification of h_1. Write P_1, P_2, P_3 for the branch points of f with branch cycles x_1, x_2, x_3, respectively. Then $\pi_0^{-1}(P_1)$ consists of two points, $\pi_0^{-1}(P_i)$ consists of three points for each $i \in \{2, 3\}$, and $\pi_0^{-1}(P)$ consists of six points for any other point P. Since $m_{h_1}(Q)$ is the same for every point $Q \in \pi_0^{-1}(P)$ over a given point P of $X_0 = \mathbb{P}^1$ by Remark 4.3, and since the list of greater than 1 values of $m_{h_1}(Q)$ is given to be $\{3, 3, 3\}$, it follows that $m_{h_1}(Q) > 1$ for all $Q \in \pi_0^{-1}(P_0)$ for exactly one point P_0 of $\mathbb{P}^1$, and that point P_0 is either P_2 or P_3. Since the argument is the same in both cases, we shall assume $P_0 = P_2$, and denote $\pi_0^{-1}(P_2) = \{Q_1, Q_2, Q_3\}$, so that $m_{h_1}(Q_i) = 3$ for $i = 1, 2, 3$, and $m_{h_1}(Q) = 1$ for any other point Q of Y.

By Lemma 2.1, the points Q_1, Q_2, Q_3 correspond to the double cosets $\langle x_2\rangle\backslash(S_\ell \wr S_3)/S_\ell^3$. Since $1, r, r^2$ is a set of representatives for these double cosets, the lemma implies that, up to reordering the points, the branch cycles of h over Q_1, Q_2, Q_3 are conjugate in S_ℓ^3 to

$$y_2^2 = (b_1, b_1, b_2^2),\ (y_2^2)^r = (b_2^2, b_1, b_1),\ (y_3^2)^{r^2} = (b_1, b_2^2, b_1),$$

respectively. Thus the branch cycles of h_1 over Q_1, Q_2, Q_3 are b_1, b_2^2, b_1, respectively, and hence these are of almost Galois type 3 with error at most $\varepsilon = \varepsilon_{\alpha,3}$. Similarly, the branch cycle of h_1 over each of the two preimages in $\pi_0^{-1}(P_1)$ is a, and the branch cycles over points in $\pi_0^{-1}(P_3)$ are c_1, c_1, c_2^2. Thus a, c_1, c_2^2 are of almost Galois type 1 with error at most $\varepsilon_{\alpha,1} \le \varepsilon$.

The remaining difficulty is the main one: showing that b_2 and c_2 are of almost Galois type. Since W is a product 1 multiset, there is a covering $\hat{f} : \hat{Y} \to \mathbb{P}^1$ whose branch cycles are the elements of W. Riemann–Hurwitz for $\hat{f}$ then gives:

$$\sum_{x\in W} (\ell - |\mathrm{orb}(x)|) \ge 2\ell - 2. \tag{8.2}$$

On the other hand, since a, b_1, c_1 are of almost Galois type with error at most ε, we have $\ell - |\mathrm{orb}(a)| < \varepsilon$, and $\ell - |\mathrm{orb}(c_1)| < \varepsilon$, and $\ell - |\mathrm{orb}(b_1)| < 2\ell/3 + \varepsilon$. We now estimate the number of orbits of b_2 and c_2. Denote by u_3 and u_6 the number of orbits of b_2 with length 3 and length 6, respectively, and by v_1 and v_2 the number of length 1 and length 2 orbits of c_2, respectively. Since c_2^2 is of almost Galois type 1 with error at most ε, we have $v_1 + 2v_2 \ge \ell - \varepsilon$ and $|\mathrm{orb}(c_2)| \ge v_1 + v_2$, the combination of which gives $\ell - |\mathrm{orb}(c_2)| \le v_2 + \varepsilon$. Similarly, since b_2^2 is of almost Galois type 3, we have $\ell - |\mathrm{orb}(b_2)| \le 2u_3 + 5u_6 + \varepsilon$. Combining the above bounds with (8.2) we get:

$$2\ell - 2 \le \sum_{x\in W} (\ell - |\mathrm{orb}(x)|) < 2u_3 + 5u_6 + v_2 + 2\ell/3 + 5\varepsilon. \tag{8.3}$$

Furthermore, one has $v_2 \le \ell/2$ and since $3u_3 + 6u_6 \le \ell$ and $u_6 \le \ell/6$ we also have $2u_3 + 5u_6 \le 2(\ell/3 - 2u_6) + 5u_6 \le 5\ell/6$. Plugging each of the two inequalities into (8.3) gives:

$$v_2 > 4\ell/3 - (2u_3 + 5u_6) - 5\varepsilon > \ell/2 - 5\varepsilon \text{ and}$$
$$2u_3 + 5u_6 > 4\ell/3 - v_2 - 5\varepsilon > 5\ell/6 - 5\varepsilon.$$

The first inequality implies that c_2 is of almost Galois type 2 with error at most 10ε. Since $u_3 \le \ell/3 - 2u_6$, the second inequality implies that $2\ell/3 + u_6 > 5\ell/6 - 5\varepsilon$ and hence $u_6 > \ell/6 - 5\varepsilon$. Thus b_2 is of almost Galois type 6 with error at most 30ε. □

The proof of Proposition 8.6 generalizes the proof of the claim, to cases where the list of $m_{h_1}(P)$ values is general and not merely $\{3, 3, 3\}$.

Proof. **Step I:** *Describing the branch cycles of h_i, $i \in I$.* Let P_j be the point of $\mathbb{P}^1$ corresponding to the branch cycle x_j, for $j \in J$, and $\tilde{h} : \tilde{X} \to \tilde{X}/K$ the natural projection. The points $Q_{j,\tau}$ in $\pi_0^{-1}(P_j)$ are in one to one correspondence with double cosets $K\tau\langle x_j\rangle$ in $K\backslash G/\langle x_j\rangle$ by Lemma 2.1, so that $(x_j^\tau)^{e_j} = z_j^\tau$ is a branch cycle of $\tilde{h}$ over $Q_{j,\tau}$ for $j \in J$. Since K_i is a point stablizer in the action of K on the i-th copy $\Delta^{(i)}$ of Δ, the monodromy group G_i of $h_i : \tilde{X}/K_i \to \tilde{X}/K$ is isomorphic to the projection of $K \le S_\Delta^I$ to $S_{\Delta^{(i)}}$, for $i \in I$. Identifying G_i with this subgroup of $S_{\Delta^{(i)}}$, the image $z_j^\tau(i) \in S_{\Delta^{(i)}}$ of $z_j^\tau \in S_\Delta^I$ is a branch cycle of h_i over $Q_{j,\tau}$, for every coset $K\tau\langle x_j\rangle \in K\backslash G/\langle x_j\rangle$, $j \in J$, and $i \in I$. Since h_i, $i \in I$ admit a pairwise genus bound $\alpha\ell$, their branch cycles $z_j(i)$, $i \in I$, $j \in J$ are all of almost Galois type with constant α by Corollary 4.4. As remarked at the end of Definition 4.6, the types $m_j = m(z_j(i))$ and errors $\varepsilon_j = \varepsilon_{\alpha,m_j}$ (resp., entry bounds N) are independent of $i \in I$, for each $j \in J$.

Step II: *Describing the sets O_j, $j \in J$.* As in the beginning of the section, let $\overline{\pi}_0 : \overline{Y} \to \mathbb{P}^1$ be the natural projection from the quotient $\overline{Y}$ of Y by a point stabilizer in the action on I, so that its monodromy group $\overline{G}$ is isomorphic to G/K equipped with its action on I. Let σ_j be the image of x_j in $\overline{G}$, and put $m := |G/K| = \deg \pi_0$. In particular, the ramification type $E_{\overline{\pi}_0}(P_j)$ coincides with the cycle structure of σ_j, for $j \in J$. Since $g_Y \le 1$ by Corollary 4.4, and since by assumption O_k consists of odd length orbits and a single orbit of length 2 for some k, we may apply Lemma 6.2 to obtain the possibilities for the ramification type $E_{\overline{\pi}_0}(P_j)$, and hence for the cycle structure of σ_j, for $j \in J$. The lemma implies that the ramification type of $\overline{\pi}_0$ is in Table 6.1. Letting $S = \{j : \sigma_j \text{ is a transposition}\}$, letting F be the multiset $\{\theta \in O_j : j \in S, |\theta| = 1\}$, and $t = |I|$, it follows that $|\theta| = e_j$ for every $\theta \in O_j$, $j \in J \setminus S$, and

(1) $|F| = 4$ with $t \le 6$ if $g_Y = 1$;
(2) $|F| = 2$ with $t \in \{3, 4\}$ if $g_Y = 0$ and $\overline{G} \not\cong S_4$; and
(3) $|F| = 3$ with $t = 4$ if $g_Y = 0$ and $\overline{G} \cong S_4$.

Step III: *Proof in the case $j \in J \setminus S$.* Let $y_j = b_j\sigma_j$, $b_j \in S_\Delta$ be a reduced form of x_j with representatives ι_θ, $\theta \in O_j$, such that $b_j(\iota_\theta) = y_{\theta,j}$, for $\theta \in O_j$, $j \in J$. Since x_j and y_j are conjugate by an element of S_Δ^I, the elements $z_j = x_j^{e_j}$ and $d_j := y_j^{e_j} \in S_\Delta^I$ are also conjugate in S_Δ^I, and hence $d_j(i)$ is of almost Galois type m_j with error at most ε_j if $m_j < \infty$ (resp., entry bound N if $m_j = \infty$), for all $i \in I$, $j \in J$. Since the orbits of σ_j are all of length $|\theta| = e_j$, we have

$$d_j(i) = \prod_{k=0}^{e_j-1} b_j^{\sigma_j^{-k}}(i) = \prod_{k=0}^{e_j-1} b_j(i^{\sigma_j^k}) = b_j(\iota_\theta) = y_{\theta,j},$$

for every $i \in \theta$, $\theta \in O_j$, $j \in J \setminus S$. Thus, $y_{\theta,j}$ is of almost Galois type $m(y_{\theta,j}) = m(d_j(\iota_\theta)) = m(z_j(\iota_\theta)) = m_j$ with error at most ε_j if $m_j < \infty$ (resp., entry bound N if $m_j = \infty$), for every $\theta \in O_j$, $j \in J \setminus S$.

Step IV: *Proof in the case $j \in S$, with m_j infinite or even.* Since the ramification of $\overline{\pi}_0$ is as in Table 6.1, in this case $e_j = 2$ and $|\theta| = 1$ or 2, for every $\theta \in O_j$. If $|\theta| = 2$, then the same argument as in step III shows that $y_{\theta,j} = b_j(\iota_\theta)$ is of almost Galois type $m(y_{\theta,j}) = m(d_j(\iota_\theta)) = m(z_j(\iota_\theta)) = m_j$ with error at most ε_j if $m_j < \infty$ (resp., entry bound N if $m_j = \infty$), for $\theta \in O_j$, $j \in S$. For $\theta \in O_j$, $j \in S$ with $|\theta| = 1$, we have $d_j(\iota_\theta) = b_j(\iota_\theta)b_j(\iota_\theta^{\sigma_j}) = b_j(\iota_\theta)^2$, and hence $b_j(\iota_\theta)^2$ is of almost Galois type m_j with error ε_j if $m_j < \infty$ (resp., entry bound N if $m_j = \infty$). If $m_j = \infty$ this implies that the number of orbits of $b_j(\iota_\theta)$ is at most N, proving the lemma in this case. Since the square of a length $k \cdot m_j$ cycle in S_ℓ is a product of cycles of length $km_j/(2, km_j)$, this square is a product of m_j-cycles if and only if $k \in \{1, 2\}$ if m_j is odd, and $k = 2$ if m_j is even. Thus, letting $n_{\theta,k}$ denote the number of length km_j orbits of $b_j(\iota_\theta)$, we have $(\ell - \varepsilon_j)/m_j \le n_{\theta,1} + 2n_{\theta,2} \le \ell/m_j$ if m_j is odd, and $(\ell - \varepsilon_j)/m_j \le 2n_{\theta,2} \le \ell/m_j$ if m_j is even. The lemma then also follows in the case where m_j is even.

Step V: *The Riemann–Hurwitz formula for the reduced multiset.* By Riemann's existence theorem there exists a covering $\hat{f} : \hat{Y} \to \mathbb{P}^1$ corresponding to the reduced product 1 multiset $y_{\theta,j}$, $\theta \in O_j$, $j \in J$. To prove that $y_{\theta,j} = b_j(\iota_\theta)$ is of almost Galois type in the remaining case where m_j is odd, $j \in S$ and $\theta \in O_j \cap F$, we bound its Riemann–Hurwitz contribution to $\hat{f}$ by

$$\begin{aligned} \ell - |\mathrm{Orb}_\Delta(y_{\theta,j})| &\le \ell - n_{\theta,1} - n_{\theta,2} = (\ell - n_{\theta,1} - 2n_{\theta,2}) + n_{\theta,2} \\ &\le \ell(1 - 1/m_j) + n_{\theta,2} + \varepsilon_j/m_j. \end{aligned} \tag{8.4}$$

Throughout the rest of the proof we use the convention $1/m_j = 0$ in case $m_j = \infty$. By steps III and IV, similarly to (8.4), we have

$$\ell - |\mathrm{Orb}_\Delta(y_{\theta,j})| \le \begin{cases} \ell(1 - 1/m_j) + \varepsilon_j/m_j & \text{if } m_j < \infty \\ \ell(1 - 1/m_j) & \text{if } m_j = \infty. \end{cases} \tag{8.5}$$

for all $\theta \in O_j$, $j \in J$ such that either $j \in J \setminus S$, or $|\theta| > 1$, or m_j is even or infinite.

Plugging (8.4) and (8.5) to the Riemann–Hurwitz formula for $\hat{f}$ gives:

$$\begin{aligned} 2\ell - 2 &\leq \sum_{j\in J}\sum_{\theta\in O_j}(\ell - |\mathrm{Orb}_\Delta(y_{\theta,j})|) \\ &\leq \sum_{j\in J}|O_j|\ell(1-\frac{1}{m_j}) + \sum_{\{j\in J:\, m_j<\infty\}}\frac{|O_j|\varepsilon_j}{m_j} + \sum_{\{j\in S:\, m_j \text{ is odd}\}}\sum_{\theta\in F\cap O_j} n_{\theta,2} \end{aligned} \tag{8.6}$$

Set $\delta_0 := \sum_{j\in J, m_j<\infty}|O_j|\varepsilon_j/m_j$ so that $\delta_0 < t\varepsilon$.

Step VI: *Separation into cases according to almost Galois types.* We claim that $n_{\theta,2} \geq \ell/(2m_j) - 3m(\varepsilon + N)$, as this would show that $y_{\theta,j}$ is of almost Galois type $2m_j$, and error at most $6mm_j(\varepsilon+N)$ for all $j\in S$, $\theta\in O_j\cap F$, completing the proof.

At first consider the case $g_Y = 1$ in which case $|F| = 4$ and $t\leq 6$. Since h_i, $i\in I$ admit a pairwise genus bound $\alpha\ell$ and $g_Y = 1$, we also have $m_j = 1$ for $j\in J$ by Corollary 4.4. Hence $\sum_{\theta\in F} n_{\theta,2} \geq 2\ell - \delta_0$ by (8.6). Since $|F|\leq 4$ and $n_{\theta,2}\leq \ell/2$, we deduce that $n_{\theta,2} > \ell/2 - \delta_0 > \ell/2 - t\varepsilon$ for $\theta\in F$. As $t\leq m$, this shows that $y_{\theta,j}$ is of almost Galois type 2 with error at most $2t\varepsilon \leq 2m\varepsilon$ for all $\theta\in F\cap O_j$, $j\in J$, as desired.

Henceforth assume $g_Y = 0$. We first modify (8.6) using the Riemann–Hurwitz formula for h_i, $i\in I$. Since $Q_{j,\tau}$ is of almost Galois type m_j under h_i, the cardinality $|E_{h_i}(Q_{j,\tau})|$ is at most $(\ell-\varepsilon_j)/m_j + \varepsilon_j$ if $m_j<\infty$ and at most N if $m_j = \infty$, for $i\in I$. Hence

$$R_{h_i}(Q_{j,\tau}) = \ell - |E_{h_i}(Q_{j,\tau})| \geq \begin{cases} \ell(1-1/m_j) - \varepsilon_j(1-1/m_j) & \text{if } m_j<\infty \\ \ell(1-1/m_j) - N & \text{if } m_j = \infty, \end{cases} \tag{8.7}$$

for every $j\in J$, and coset $\tau\in K\backslash G/\langle x_j\rangle$. Since $g_{Y_i}\leq \alpha+1$ by Remark 4.5, and since $|\pi_0^{-1}(P_j)| = m/e_j$ as π_0 is Galois, the Riemann-Hurwitz formula for h_i and (8.7) give:

$$\begin{aligned} &2(\ell+\alpha) \geq 2(\ell + g_{Y_i} - 1) = \sum_{j\in J} R_{h_i}(\pi_0^{-1}(P_j)) \geq \sum_{j\in J}\frac{m}{e_j}\ell(1-\frac{1}{m_j}) - \delta_1, \text{ where} \\ &\delta_1 := \sum_{j\in J:\, m_j<\infty}\frac{m}{e_j}\varepsilon_j(1-1/m_j) + \sum_{j\in J:\, m_j=\infty}\frac{m}{e_j}N. \end{aligned} \tag{8.8}$$

Note that since the ramification of π_0 is as in Table 6.1 and $g_Y = 0$, one has $\sum_{j\in J}(1/e_j) < 2$ and hence also $\delta_1 < 2m(\varepsilon+N)$. The combination of (8.6) and (8.8) then gives:

$$\begin{aligned} \sum_{\{j\in S:\, m_j \text{ is odd}\}}\sum_{\theta\in F\cap O_j} n_{\theta,2} &\geq 2\ell - 2 - \sum_{j\in J}|O_j|\ell(1-\frac{1}{m_j}) - \delta_0 \\ &\geq -2 - 2\alpha - \delta_0 - \delta_1 + \sum_{j\in J}(\frac{m}{e_j} - |O_j|)\ell(1-\frac{1}{m_j}) \\ &= \sum_{j\in J}(\frac{m}{e_j} - |O_j|)\ell(1-\frac{1}{m_j}) - \delta_2, \end{aligned} \tag{8.9}$$

where $\delta_2 := 2 + 2\alpha + \delta_0 + \delta_1$. Note that as $\delta_0 < t\varepsilon$ and $\delta_1 < 2m(\varepsilon + N)$, one has $\delta_2 < 2(\alpha+1) + t\varepsilon + 2m(\varepsilon + N)$. As $2(\alpha+1) < \varepsilon$ and $t+1 < m$ in all cases in Table 6.1 with $t > 2$, one further has $\delta_2 < 2(\alpha+1) + t\varepsilon + 2m(\varepsilon+N) < 3m(\varepsilon+N)$.

We estimate (8.9) for each of the possibilities for the multiset:

$$M_h := \{m_j = m_{h_i}(Q_{j,\tau})\colon m_j > 1, \tau \in K\backslash G/\langle x_j\rangle, j \in J\},$$

which as remarked earlier is independent of the choice of $i \in I$. Note that as $m_j = m_{h_i}(Q_{j,\tau})$ is also independent of τ, it appears in M_h at least $|K\backslash G/\langle x_j\rangle| = m/e_j$ times. By going over ramification types for π_0 corresponding to the possibilities for the ramification of $\overline{\pi}_0$ in Table 6.1, Corollary 4.4 shows that $M_h = \{m_\eta\colon \tau \in K\backslash G/\langle x_\eta\rangle\}$ for some $\eta \in J$, and there are three possibilities:

(I1) $M_h = \{\infty,\infty\}$, and $E_{\pi_0}(P_\eta) = [t,t]$, and $E_{\overline{\pi}_0}(P_\eta) = [t]$ for $t = 3$ or 4;
(F1) $M_h = \{3,3,3\}$, and $E_{\pi_0}(P_\eta) = [2^3]$, and $E_{\overline{\pi}_0}(P_\eta) = [2,1]$ for $t = 3$;
(F2) $M_h = \{2,2,2,2\}$, and $E_{\pi_0}(P_\eta) = [2^4]$, and $E_{\overline{\pi}_0}(P_\eta)$ is $[2,1^2]$ or $[2^2]$ for $t = 4$.

In all three cases $\overline{G}$ is Dihedral of order $2t$. Note that as $E_{\overline{\pi}_0}(P_\eta)$ is the cycle structure of σ_j, we have $\eta \notin S$ in case (I1) and $\eta \in S$ in case (F1). It follows that $m_j = 1$ for $j \in J\backslash\{\eta\}$, and $|F| = 2$ as $\overline{G} \not\cong S_4$. Thus (8.9) reduces to

$$\sum_{\{j\in S\colon\ m_j \text{ is odd}\}} \sum_{\theta\in F\cap O_j} n_{\theta,2} \geq \left(\frac{m}{e_\eta} - |O_\eta|\right)\left(1 - \frac{1}{m_\eta}\right)\ell - \delta_2. \tag{8.10}$$

In case (I1), $\eta \notin S$, $m_\eta = \infty$, $e_\eta = m/2$, and x_η has one orbit, giving $|O_\eta| = 1$. Hence $\sum_{\theta\in F} n_{\theta,2} \geq \ell - \delta_2$ by (8.10). Since in addition $n_{\theta,2} \leq \ell/2$ for each $\theta \in F$ and $|F| = 2$, we have $n_{\theta,2} \geq \ell/2 - \delta_2$ for all $\theta \in F$, as desired.

In case (F1), $\eta \in S$, $m_\eta = 3$, $m = 6$, $e_\eta = 2$, and $|O_\eta| = 2$. Hence (8.10) gives $\sum_{\theta\in F} n_{\theta,2} \geq 2\ell/3 - \delta_2$. As $|S| = 2$, write $S = \{\eta,\mu\}$. In this case, σ_η and σ_μ both have a single fixed point, denoted by θ_1 and θ_2, respectively. Since $n_{\theta_1,2} \leq \ell/(2m_\eta) = \ell/6$ and $n_{\theta_2,2} \leq \ell/(2m_\mu) = \ell/2$, the conclusion from (8.10) is that $n_{\theta_1,2} \geq \ell/6 - \delta_2 = \ell/(2m_\eta) - \delta_2$ and $n_{\theta_2,2} \geq \ell/2 - \delta_2 = \ell/(2m_\mu) - \delta_2$, as desired.

In case (F2), $m = 8$, $m_\eta = 2$, $e_\eta = 2$, and $S = \{\mu\}$ for some $\mu \in J$. Moreover, the case $\mu = \eta$ is covered in Step IV, as then $m_\mu = m_\eta$ is even. Hence, we may assume $\mu \neq \eta$, in which case $|O_\eta| = 2$. Then (8.10) reduces to $\sum_{\theta\in F} n_{\theta,2} \geq \ell - \delta_2$ and by definition $n_{\theta,2} \leq \ell/2$ for $\theta \in F$. Thus in this case $n_{\theta,2} \geq \ell/2 - \delta_2$ for $\theta \in F$, as desired. □

Proof of Proposition 7.2. Let $x_1,\dots,x_r \in G$ be a product 1 tuple corresponding to f, and P_j the branch point corresponding to x_j. Put $J = \{1,\dots,r\}$ and let R_π be the sum of contributions $R_\pi(f^{-1}(P))$ over all points P of $\mathbb{P}^1$. We note that since h_i, $i \in I$ admit a pairwise genus bound $\alpha\ell$, and since the number of branch points of π_0 is at most four as $g_Y \leq 1$, it follows by Corollary 4.4 that the number of branch points r is bounded by a constant $B = B_{\alpha,t} > 0$, depending only on α and t.

Let S be the set of all $j \in J$ such that the action of x_j on I consists of odd length orbits and a single orbit of length 2. Proposition 5.1.(2) implies that $R_\pi(f^{-1}(P_j)) < E_0\ell^{t-2}$ for all $j \in J \setminus S$, where $E_0 = E_{0,t} > 0$ is a constant depending only on t. In particular, if $S = \emptyset$, we have $R_\pi < E_3\ell^{t-2}$ for a constant $E_3 := BE_0$, which depends only on α and t.

Henceforth, assume $S \neq \emptyset$. Recall that as in the beginning of the section, the natural projection $\overline{\pi}_0 : \overline{Y} \to \mathbb{P}^1$ from the quotient $\overline{Y}$ by a point stabilizer in the action of G on I, has Galois closure π_0, and monodromy group G/K equipped with its action on I. In particular, the ramification type $E_{\overline{\pi}_0}(P_j)$ coincides with the cycle structure of the image σ_j in G/K of x_j, for $j \in J$. Since $S \neq \emptyset$ and $g_Y \leq 1$ by Corollary 4.4, the ramification type of $\overline{\pi}_0$ appears in Table 6.1 by Lemma 6.2, and hence S consists of those $j \in J$ for which x_j acts on I as a transposition. Hence, there exists a product 1 tuple $x_1', \ldots, x_r'$ which satisfies condition (LG$_Y$) and whose ramification type is the same as that of f, by Lemma 8.4. We can therefore apply Proposition 8.2 and deduce that there exists a reduced tuple $y_{\theta,j}$ where θ runs through the set O_j of orbits of x_j' on I, for $j \in J$.

Let e_j be the order of the image of x_j' in G/K, and $z_j := (x_j')^{e_j} \in S_\Delta^I$, for $j \in J$. By Proposition 8.6, $y_{\theta,j}$ is of almost Galois type $m(y_{\theta,j}) = 2m(z_j(\iota_\theta)) < \infty$ with error at most $\hat{\varepsilon}_{\alpha,t}$ depending only on α and t, if $j \in S$ and $\theta \in O_j$ has length 1. Since $t \geq 3$ and x_j' acts as a transposition on I, each x_j' has an orbit $\tilde{\theta} \in O_j$ of length 1 for every $j \in S$, so that $m(y_{\theta,j})$ is even. Hence, Corollary 5.3 implies that there exists a constant $E_4 = E_{4,t,\alpha}$ such that $R_\pi(f^{-1}(P_j)) < E_4 \ell^{t-2}$ for all $j \in S$. In total we have $R_\pi < E_2 \ell^{t-2}$ for $E_2 \geq B \cdot \max\{E_3, E_4\}$. ☐

9. Theorem 7.1: The case $t = 2$

9.1. **Relating g_X and g_{Y_1}.** We use the setup of §2.7 with $t = 2$, $X_0 = \mathbb{P}^1$, and write $I = \{1, 2\}$, so that $f : X \to \mathbb{P}^1$ is an indecomposable covering with monodromy group $G \leq S_\Delta \wr S_2$ of product type, and $\tilde{X}$ is its Galois closure. We associate to f the coverings $h_i : Y_i \to Y$, where $Y_i := \tilde{X}/K_i$, $Y := \tilde{X}/K$, $K := G \cap S_\Delta^2$, and K_i a point stabilizer in the action of K on the i-th copy of Δ, $i = 1, 2$. Recall that by Remark 2.15, the fiber product of h_1 and h_2 is Y, and each of them is indecomposble.

The following lemma gives an explicit formula for the genus of X in terms of the ramification of h_i, $i = 1, 2$. The proof of Theorem 7.1 for $t = 2$, given in this section, relies on this formula and estimates of it. As in Setup 2.7, let $\pi_0 : Y \to X_0$ be the natural projection, so that $\deg \pi_0 = [G : K] = 2$ by Lemma 2.9. Fix $\sigma \in G \setminus K$, and let $\overline{\sigma} : Y \to Y$ be the automorphism it induces, cf. Remark 2.15.(2). Note that $\overline{\sigma}$ is an involution and $\pi_0(P^{\overline{\sigma}}) = \pi_0(P)$.

Lemma 9.1. *Write the points of Y as a disjoint union $R \cup S \cup S^{\overline{\sigma}}$, where R is the set of ramification points of π_0. Then*

$$4g_X - 4 = 2\ell(g_{Y_1} - 1) + \sum_{P \in S} S_{h_1,h_2}(P) + \sum_{P \in R} S_{h_1}(P) - \sum_{P \in R} |\{r \in E_{h_1}(P) \,|\, r \text{ is odd}\}|,$$

where $S_{h_1}(P) := \sum_{r_1, r_2 \in E_{h_1}(P)} (r_1 - (r_1, r_2))$, and

$$S_{h_1,h_2}(P) := \sum_{r \in E_{h_1}(P), s \in E_{h_2}(P)} (r + s - 2(r, s)).$$

Proof. Let $h_1^2 : Z \to Y_2$ be the natural projection, P a point of Y, and $p := \pi_0(P)$. As in Setup 2.7, we have $E_{h_2}(P^{\overline{\sigma}}) = E_{h_1}(P)$ for every point P of Y. Note $P^{\overline{\sigma}} = P$ if and only if $P \in R$. Thus, Abhyankar's Lemma 2.2 implies:

$$R_{h_1^2}(h_2^{-1}(p)) = R_{h_1^2}(P) = \sum_{r \in E_{h_1}(P), s \in E_{h_2}(P)} (r - (r,s)) = \sum_{r_1, r_2 \in E_{h_1}(P)} (r_1 - (r_1, r_2))$$

if $P \in R$, and

$$\begin{aligned} R_{h_1^2}(h_2^{-1}(p)) =& R_{h_1^2}(P) + R_{h_1^2}(P^{\overline{\sigma}}) \\ =& \sum_{r \in E_{h_1}(P), s \in E_{h_2}(P)} (r - (r,s)) + \sum_{s \in E_{h_1}(P^{\overline{\sigma}}), r \in E_{h_2}(P^{\overline{\sigma}})} (s - (r,s)) \\ =& \sum_{r \in E_{h_1}(P), s \in E_{h_2}(P)} (r + s - 2(r,s)), \end{aligned}$$

if $P \in S$. Thus the Riemann-Hurwitz formula for h_1^2 gives:

$$2(g_Z - 1) = 2\ell(g_{Y_2} - 1) + \sum_{P \in R} S_{h_1}(P) + \sum_{P \in S} S_{h_1,h_2}(P)$$

Substituting $g_{Y_2} = g_{Y_1}$ as $Y_2 \cong Y_1$ by Remark 2.15.(2), and replacing the left hand side using the Riemann–Hurwitz formula for the natural projection $\pi : Z \to X$ gives

$$4(g_X - 1) + \sum_{p \in X_0(\mathbb{K})} R_\pi(f^{-1}(p)) = 2\ell(g_{Y_1} - 1) + \sum_{P \in R} S_{h_1}(P) + \sum_{P \in S} S_{h_1,h_2}(P).$$

The claim follows from the latter equality since $R_\pi(f^{-1}(p)) = |\{r \in E_{h_1}(p) \mid r \text{ is odd}\}|$ by Proposition 5.1.(3) and Remark 2.16, for every point p of X_0. □

We use the following estimates on S_{h_1} and S_{h_1,h_2}. We denote by $O_\alpha(1)$ a constant depending only on α, and write $X = Y + O_\alpha(1)$ to denote that $|X - Y|$ is bounded by a constant depending only on α.

Lemma 9.2. *Assume $h_i : Y_i \to Y$, $i = 1,2$, admit a pairwise genus bound $\alpha\ell$ for some integer $\alpha > 0$, and P a point of Y of almost Galois type $m_P := m_{h_1}(P) = m_{h_2}(P)$. If $m_P < \infty$, then $S_{h_1,h_2}(P) = S_{h_1}(P) + S_{h_2}(P) + O_\alpha(1)$, and*

$$S_{h_1}(P) = \begin{cases} \ell R_{h_1}(P) + O_\alpha(1) \text{ if } m_P = 1; \\ \ell[\frac{\ell}{2} - |E_{h_1}(P)| + |\{r \in E_{h_1}(P) \mid \ r \text{ is odd}\}|] + O_\alpha(1) \text{ if } m_P = 2; \\ \ell[\frac{\ell}{3} - |E_{h_1}(P)| + \frac{4}{3}|\{r \in E_{h_1}(P) \mid (r,3) = 1\}|] + O_\alpha(1) \text{ if } m_P = 3; \\ \ell[\frac{\ell}{4} - |E_{h_1}(P)| + |\{r \in E_{h_1}(P) \mid r \equiv 2\,(4)\}| + \frac{3}{2}|\{\text{odd } r \in E_{h_1}(P)]| \\ +O_\alpha(1) \text{ if } m_P = 4; \\ \ell\left[\frac{\ell}{6} - |E_{h_1}(P)| + |\{r \in E_{h_1}(P) \mid r \equiv 3\,(6)\}| + \frac{4}{3}|\{r \in E_{h_1}(P) \mid r \equiv \pm 2\,(6)\}| \right. \\ \left. + \frac{5}{3}|\{r \in E_{h_1}(P) \mid r \equiv \pm 1\,(6)\}|\right] + O_\alpha(1) \text{ if } m_P = 6. \end{cases}$$

If $m_P = \infty$, then $S_{h_1,h_2}(P) \geq (u+v)\ell - 2\ell \min(u,v)$, where $u := |E_{h_1}(P)|$ and $v := |E_{h_2}(P)|$. Moreover, if $u \leq v \leq 3$, and the greatest common divisor of all entries in $E_{h_1}(P)$ and $E_{h_2}(P)$ is 1, then $S_{h_1,h_2}(P) \geq (u+v-2\min(u,v)-\xi)\ell$ for an absolute constant $\xi > 0$.

The proof relies on the following lemma:

Lemma 9.3. *Let $r_1, ..., r_u, s_1, ..., s_v$ be positive integers such that $\sum_{i=1}^u r_i = \sum_{j=1}^v s_j = \ell$, and $u \leq v$. Let $S := \sum_{i=1}^u \sum_{j=1}^v (r_i + s_j - 2(r_i, s_j))$. Then*

1. *$S \geq \ell(v-u)$;*
2. *There exists an absolute constant $\xi > 0$ such that if $v \leq 3$, and $\ell \geq (42v)^v$, and $(r_1, \dots, r_u, s_1, \dots, s_v) = 1$, then $S \geq \ell(v-u+\xi)$.*
3. *If $u = v = 2$, $\{r_1, r_2\} \cap \{s_1, s_2\} = \emptyset$, and $(r_1, r_2, s_1, s_2) = 1$, then $S > 5\ell/2 - 5$.*

Remark 9.4. Our proof of part (2) generalizes to a statement for arbitrary v. Namely, there exists a constant $\xi_v > 0$, depending only on v, such that for every ℓ sufficiently large compared to v, and positive integers $r_1, \dots, r_u$ and $s_1, \dots, s_v$ for $u \leq v$ such that $\sum_{i=1}^u r_i = \sum_{j=1}^v s_j = \ell$ and $(r_1, \dots, r_u, s_1, \dots, s_v) = 1$, the above sum S is at least $\ell(v-u+\xi)$. For this, one needs to show that for $v \geq 1$, there exists a constant $0 < D_v < 1$ such that every sum of at most v divisors of ℓ does not lie in the interval $((1-D_v)\ell, \ell)$. One then replaces D with D_v throughout the proof.

Proof of Lemma 9.3. (1) Since $(r_i, s_j) \leq s_j$ for all i, j, since $\sum_{i=1}^u \sum_{j=1}^v s_j = u\ell$ and $\sum_{i=1}^u \sum_{j=1}^v r_i = v\ell$, we have $S \geq \ell(v+u) - 2\sum_{i=1}^u \sum_{j=1}^v s_j = \ell(v-u)$, as desired.

(2) Now assume $v \leq 3$, and $(r_1, \dots, r_u, s_1, \dots, s_v) = 1$. Putting $D = 1/42$, a straightforward check shows that for any w with $0 \leq w \leq 3$, the sum of w divisors of ℓ does not lie in the interval $((1-D)\ell, \ell)$. Note that the sum

$$T := \frac{S - \ell(v-u)}{2} = \sum_{i=1}^u \sum_{j=1}^v \big(s_j - (r_i, s_j)\big)$$

is at least $s_j - (r_i, s_j)$ for any prescribed i, j. Put $\xi := D/v$. If there exist i, j such that $s_j \geq D\ell/v$ and s_j doesn't divide r_i, then (r_i, s_j) is a proper divisor of s_j and hence is at most $s_j/2$, so that

$$T \geq s_j - (r_i, s_j) \geq \frac{s_j}{2} > \frac{D}{2v}\ell > \frac{\xi}{2}\ell, \text{ and hence } S \geq \ell(v-u+\xi).$$

Now suppose that $\ell > (v/D)^v$ and that every s_j which is at least $D\ell/v$ divides all r_i, $i = 1, \dots, u$. Let $J := \{1, \dots, v\}$, let U_1 be the multiset $\{s_j : j \in J, \text{ and } s_j \geq D\ell/v\}$, and let U_2 be the multiset $\{s_j : j \in J, \text{ and } s_j < D\ell/v\}$. Since $s_1 + \ldots + s_v = \ell$, the biggest s_j is at least ℓ/v, which is at least $D\ell/v$ since $D \leq 1$, so that U_1 contains an element which is at least ℓ/v. Each $s \in U_1$ divides every r_i, and hence divides $\ell = \sum_{i=1}^u r_i$. Thus the elements of U_1 are divisors of ℓ, and

$$\gcd(U_1) = \frac{\ell}{\operatorname{lcm}(\{\ell/s : s \in U_1\})} \geq \frac{\ell}{\prod_{s \in U_1} \frac{\ell}{s}} \geq \frac{\ell}{\prod_{s \in U_1} \frac{v}{D}} \geq \frac{\ell}{(v/D)^v} > 1.$$

Since $\gcd(U_1)$ is nontrivial and divides $(r_1, ..., r_u)$, the assumption $(r_1, \ldots, r_u, s_1, \ldots, s_v) = 1$ implies that U_2 is nonempty. Thus U_1 is a multiset of divisors of ℓ whose sum is less than ℓ; since $|U_1| < v$, it follows that the sum of the elements of U_1 is at most $(1-D)\ell$, whence the sum of the elements of U_2 is at least $D\ell$. Here U_2 is a multiset of integers $1 \le z < D\ell/v$, and $|U_2| \le v$, so the sum of the elements of U_2 is less than $D\ell$, yielding the desired contradiction.

(3) Now let $u = v = 2$. Let $S_i := (r_1, s_i) + (r_2, s_i)$, for $i = 1, 2$. Without loss of generality assume s_2 is the maximum in $\{r_1, r_2, s_1, s_2\}$. Since $(r_1, r_2, s_1, s_2) = 1$ and $s_2 = r_1 + r_2 - s_1$ it follows that $(r_1, r_2, s_1) = 1$. Hence (r_1, s_1) and (r_2, s_1) are coprime. As both divide s_1, their product also divides s_1. It follows that either $S_1 \le s_1/2 + 2$, or $s_1 \mid r_j$ for some $j \in \{1, 2\}$ in which case $S_1 = s_1 + 1$. As $s_2 > r_j$, $j = 1, 2$, the same argument gives $S_2 \le s_2/2 + 2$. Thus $S_1 + S_2 \le s_1 + s_2/2 + 3 = \ell + 3 - s_2/2$. As $s_2 > (\ell+3)/2$, we get

$$S = 4\ell - 2(S_1 + S_2) \ge 4\ell - 2(\ell + 3) + s_2 > (5\ell - 9)/2.$$

□

Proof of Lemma 9.2. If $m_P = \infty$, we may assume without loss of generality that $|E_{h_2}(P)| \ge |E_{h_1}(P)|$. The assertion then follows immediately from Lemma 9.3.(1-2) by writing $E_{h_1}(P) = [r_1, \ldots, r_u]$ and $E_{h_2}(P) = [s_1, \ldots, s_v]$.

Now assume m_P is finite. Since there are $O_\alpha(1)$ entries of h_2 that are different from m_P, all of which are bounded by $O_\alpha(1)$, we have

$$\begin{aligned}
S_{h_1,h_2}(P) = & \sum_{\substack{r\in E_{h_1}(P), s\in E_{h_2}(P) \\ s=m_P}} (r + m_P - 2(r, m_P)) \\
& + \sum_{\substack{r\in E_{h_1}(P), s\in E_{h_2}(P), \\ r=m_P}} (s + m_P - 2(s, m_P)) + \sum_{\substack{r\in E_{h_1}(P), s\in E_{h_2}(P), \\ r,s\neq m_P}} (r + s - 2(r, s)) \\
=& \frac{\ell}{m_P} \sum_{r\in E_{h_1}(P)} (r + m_P - 2(r, m_P)) + \frac{\ell}{m_P} \sum_{s\in E_{h_2}(P)} (s + m_P - 2(s, m_P)) + O_\alpha(1) \\
=& S_{h_1}(P) + S_{h_2}(P) + O_\alpha(1).
\end{aligned}$$

The estimates for $S_{h_1}(P)$ are proved in [26, Lemma 10.3].

□

Finally, we state some estimates from [26, Proof of Proposition 10.1]:

Lemma 9.5. *There exist constants $c_3 > 0$ and d_3 such that for every indecomposable covering $h_1 : Y_1 \to \mathbb{P}^1$ with nonabelian monodromy and for which every point P of Y is of almost Galois type $m_{h_1}(P)$, the following property holds. Let $\ell := \deg h_1$, and $M_{h_1} := \{m_{h_1}(P) > 1 \mid P \in \mathbb{P}^1(\mathbb{K})\}$.*

(I1) *If $M_{h_1} = \{\infty, \infty\}$ and $2\ell(g_{Y_1} - 1) + \sum_{P\in\mathbb{P}^1(\mathbb{K})} S_{h_1}(P) < c_3\ell - d_3$ then the ramification type of h_1 is $[\ell]$, $[a, \ell - a], [2, 1^{\ell-2}]$ with $(a, \ell) = 1$;*

TABLE 9.1. Non-occurring ramification types for primitive groups $A_\ell^2 \leq G \leq S_\ell \wr S_2$ of product type.

I1A.N1	$([\ell],[\ell]),([2,1^{\ell-2}],[2,1^{\ell-2}]),s,s$
I1A.N2	$([\ell],[\ell]),([2^2,1^{\ell-4}],[1^\ell])s,s.$
F4.N1	$s,s,s,([1^{\ell-4},2^2],1)s$
F4.N2	$s,s,s,([1^{n-3},3],1)s$

(I2) *If* $M_{h_1} = \{\infty, 2, 2\}$ *and* $2\ell(g_{Y_1}-1)+\sum_{P\in\mathbb{P}^1(\mathbb{K})} S_{h_1}(P) - \ell < c_3\ell - d_3$ *then the ramification type of* h_1 *is one of the types I2.1-I2.15 in* [26, Table 4.1]*;*
(F1) *If* $M_{h_1} = \{2,2,2,2\}$ *and* $2\ell(g_{Y_1}-1)+\sum_{P\in\mathbb{P}^1(\mathbb{K})} S_{h_1}(P) - 2\ell < c_3\ell - d_3$ *then the ramification type of* h_1 *is one of the types F1.1-F1.9 in* [26, Table 4.1]*;*
(F3) *If* $M_{h_1} = \{2,4,4\}$ *and* $2\ell(g_{Y_1}-1)+\sum_{P\in\mathbb{P}^1(\mathbb{K})} S_{h_1}(P) - \ell < c_3\ell - d_3$ *then the ramification type of* h_1 *is one of the types F3.1-F3.3 in* [26, Table 4.1].

Proof. Estimating the terms $S_{h_1}(P)$ in each case using Lemma 9.2, the inequality

$$2\ell(g_{Y_1}-1)+\sum_{P\in\mathbb{P}^1(\mathbb{K})} S_{h_1}(P) < c_3\ell - d_3$$

in case (I1) coincides with [26, (10.3)] as the left hand side of the latter is bounded by $c_3\ell - d_3$ for some constants $c_3, d_3 > 0$, depending only on α. Similarly, the inequality $2\ell(g_{Y_1}-1)+\sum_{P\in\mathbb{P}^1(\mathbb{K})} S_{h_1}(P)-\ell < c_3\ell-d_3$ in case (I2) coincides with [26, (10.6)] combined with $g_{X_2} < c_3\ell - d_3$, the inequality $2\ell(g_{Y_1}-1)+\sum_{P\in\mathbb{P}^1(\mathbb{K})} S_{h_1}(P) - 2\ell < c_3\ell - d_3$ in case (F1) coincides with [26, (10.10)] with $g_X < c_3\ell - d_3$, and $2\ell(g_{Y_1}-1)+\sum_{P\in\mathbb{P}^1(\mathbb{K})} S_{h_1}(P) - \ell < c_3\ell - d_3$ in case (F3) coincides with [26, (10.14)] for $g_X < c_3\ell - d_3$. The proof of [26, Proposition 10.1] determines all ramification types of indecomposable coverings h_1 satisfying these inequalities. □

We shall also use the following lemma to show that certain ramification types do not correspond to an indecomposable covering:

Lemma 9.6. *There is no product* 1 *tuple that generates a primitive group and whose ramification corresponds to types I1A.N1, I1A.N2, F4.N1, F4.N2 in Table 9.1.*

The proof of Lemma 9.6 is given in Section 10.

9.2. **Proof of Theorem 7.1 for** $t = 2$**.** We will show that there exist sufficiently small $c_2 > 0$ and sufficiently large d_2 such that the condition $g_X < c_2\ell - d_2$ forces the ramification of f to appear in Table 3.1. Let $K := G \cap S_\ell^2$ and H be a point stabilizer of G. Let $\pi_0 : Y \to \mathbb{P}^1$ and $\pi : Z \to X$ be the natural projections from $Y := \tilde{X}/K$ and $Z := \tilde{X}/(H\cap K)$, where $\tilde{X}$ is the Galois closure of f. Recall that the coverings h_1 and h_2 are indecomposable by Remark 2.15.(3) as G is primitive.

Write the branch points of f as a disjoint union of three sets $R \cup S \cup S^{\overline{\sigma}}$ for $\overline{\sigma} \in G \setminus K$, where R is the set of branch points of π_0. Lemma 9.1 then gives

$$4g_X - 4 = 2\ell(g_{Y_1} - 1) + \sum_{P \in R} S_{h_1}(P) + \sum_{P \in S} S_{h_1,h_2}(P) - \sum_{P \in R} |\{\text{odd } r \in E_{h_1}(P)\}|. \tag{9.1}$$

Note that since $E_{h_2}(P) = E_{h_1}(P^{\overline{\sigma}})$ by Remark 2.15.(2), we have $S_{h_2}(P) = S_{h_1}(P^{\overline{\sigma}})$ for all $P \in S$. Also note that $S_{h_1,h_2}(P) = S_{h_1}(P) + S_{h_2}(P) + O_\alpha(1)$ for $P \in S$ with $m_{h_i}(P) < \infty$, and that the number of branch points of h_1 is bounded by Corollary 4.4. Thus, (9.1) gives

$$\begin{aligned} 4g_X - 4 = 2\ell(g_{Y_1} - 1) &+ \sum_{P \in R, m_{h_1}(P) = \infty} S_{h_1}(P) + \sum_{P \in S, m_{h_1}(P) = \infty} S_{h_1,h_2}(P) \\ &+ \sum_{P \in Y(\mathbb{K}), m_{h_1}(P) < \infty} S_{h_1}(P) - \sum_{P \in R} |\{\text{odd } r \in E_{h_1}(P)\}| + O_\alpha(1). \end{aligned} \tag{9.2}$$

Since h_i, $i \in I$ admit a pairwise genus bound $\alpha\ell$, Corollary 4.4 implies that $g_Y \leq 1$. Hence, $|R| = \sum_{p \in X_0(\mathbb{K})} R_{\pi_0}(p) = 2(g_Y + 1)$ is 2 if $g_Y = 0$ and 4 if $g_Y = 1$, by the Riemann–Hurwitz formula for π_0.

First assume $g_Y = 1$ and hence that $|R| = 4$, and that $m_{h_1}(P) = 1$ for every point P of Y, by Corollary 4.4. By Lemma 9.2, we have $S_{h_1}(P) = \ell R_{h_1}(P) + O_\alpha(1)$ for every point P of Y. Letting $R_{h_1} := \sum_{P \in Y(\mathbb{K})} R_{h_1}(P)$, we have in addition $2(g_{Y_1} - 1) = R_{h_1}$ for P in Y_1 by the Riemann–Hurwitz formula for h_1. Thus (9.2) gives:

$$4(g_X - 1) \geq 2\ell(g_{Y_1} - 1) + \ell R_{h_1} - 4\ell + O_\alpha(1) = 2\ell(R_{h_1} - 2) + O_\alpha(1).$$

Thus, we have $g_X > \ell/2 - O_\alpha(1)$ if $R_{h_1} > 2$. Hence by requiring $c_2 \leq 1/2$ and taking d_2 to be sufficiently large, we may ssume $R_{h_1} \leq 2$. Note that R_{h_1} is even since $R_{h_1} = 2(g_{Y_1} - 1)$, so that $R_{h_1} = 0$ or 2. If $R_{h_1} = 0$, then h_1 is a morphism of genus 1 curves and hence $\mathrm{Mon}(h_1)$ is abelian [31, Theorem 4.10(c)], contradicting the nonsolvability of G. Thus, $R_{h_1} = 2$ and the ramification type of h_1 is either $[2, 1^{\ell-2}]^2$, or $[3, 1^{\ell-3}]$, or $[2, 2, 1^{\ell-4}]$. By Remark 2.16, the ramification types of f corresponding to such h_1 appear as the F4 types in Tables 3.1 and 9.1. Types I1A.N1, I1A.N2, F4.N1, and F4.N2 in Table 9.1 do not correspond to an indecomposable covering by Lemma 9.6.

Henceforth assume $g_Y = 0$ and hence that $R = \{P_1, P_2\}$ for two points P_1, P_2 of Y. We analyze the possibilities for the multiset $M_{h_1} := \{m_{h_1}(P) > 1 \,|\, P \in Y(\mathbb{K})\}$. Since $|R| = 2$ and since $m_{h_1}(P) = m_{h_1}(P^{\overline{\sigma}})$ for every $P \in S$ by Remark 4.3, Corollary 4.4 gives the following possibilities for M_{h_1}:

- (I1A) $M_{h_1} = \{\infty, \infty\}$, $m_{h_1}(P_i) = 1$, $i = 1, 2$;
- (I1B) $M_{h_1} = \{\infty, \infty\}$, $m_{h_1}(P_i) = \infty$, $i = 1, 2$;
- (I2) $M_{h_1} = \{\infty, 2, 2\}$, $\{m_{h_1}(P_1), m_{h_1}(P_2)\} = \{1, \infty\}$;
- (F1A) $M_{h_1} = \{2, 2, 2, 2\}$, $m_{h_1}(P_i) = 1$, $i = 1, 2$;
- (F1B) $M_{h_1} = \{2, 2, 2, 2\}$, $m_{h_1}(P_i) = 2$, $i = 1, 2$;
- (F2) $M_{h_1} = \{3, 3, 3\}$, $\{m_{h_1}(P_1), m_{h_1}(P_2)\} = \{1, 3\}$;
- (F3) $M_{h_1} = \{2, 4, 4\}$, $\{m_{h_1}(P_1), m_{h_1}(P_2)\} = \{1, 2\}$.

We estimate the right hand side of (9.2) in each of these cases using Lemma 9.2. In each case we require conditions on c_2, and d_2. The theorem then follows by taking the minimal (resp., maximal) value of c_2 (resp., d_2) among all cases.

Case I1A: Let P_3 be a point of Y such that $m_{h_1}(P_3) = \infty$, and set $P_4 = P_3^{\overline{\sigma}}$, so that $m_{h_1}(P_4) = \infty$. Set $u = u_{P_3} := |E_{h_1}(P_3)|$ and $v = v_{P_4} := |E_{h_2}(P_3)|$ which equals $|E_{h_1}(P_4)|$ by Remark 2.15.(2). By Lemma 9.2, equality (9.2) gives:

(9.3)
$$4(g_X - 1) \geq 2\ell(g_{Y_1} - 1) + \ell(u+v) - 2\ell\min(u,v) + \xi_h\ell + \ell \sum_{P\in Y(\mathbb{K})\setminus\{P_3,P_4\}} R_{h_1}(P) - 2\ell + O_\alpha(1).$$

where we set ξ_h to be the constant ξ from Lemma 9.2 if $\max\{u, v\} \leq 3$ and the greatest common divisors of all entries in $E_{h_1}(P_3)$ and $E_{h_2}(P_3)$ is 1, and set $\xi_h = 0$ otherwise. On the other hand the Riemann-Hurwitz formula for h_1 gives

$$\sum_{P\in Y(\mathbb{K})\setminus\{P_3,P_4\}} R_{h_1}(P) = 2(g_{Y_1} - 1) + u + v. \tag{9.4}$$

Substituting (9.4) into (9.3) gives:

$$4(g_X - 1) \geq 4\ell(g_{Y_1} - 1) + 2\ell(u + v) - 2\ell\min(u, v) + \xi_h\ell - 2\ell + O_\alpha(1).$$

For $0 < c_2 < \xi/4$ and sufficiently large d_2, the condition $g_X < c_2\ell - d_2$ forces

$$0 \geq 4(g_{Y_1} - 1) + 2(u + v) - 2\min(u, v) + \xi_h - 2,$$

or equivalently

$$\max(u, v) \leq 3 - 2g_{Y_1} - \xi_h/2. \tag{9.5}$$

In particular $g_{Y_1} \leq 1$. First, assume $g_{Y_1} = 1$ so that $u = v = 1$ by (9.5). Then (9.4) gives $\sum_{P\in Y(\mathbb{K})\setminus\{P_3,P_4\}} R_{h_1}(P) = 2$. Thus the ramification type of h_1 is either $[\ell], [\ell], [3, 1^{\ell-3}]$, or $[\ell], [\ell], [2, 1^{\ell-2}]$ twice, or $[\ell], [\ell], [2, 2, 1^{\ell-4}]$. By Remark 2.16, these correspond to the ramification types of f appearing as types I1A.1-3 and I1A.N1-2 in Tables 3.1 and 9.1. By Lemma 9.6, types I1A.N1-2 do not correspond to an indecomposable covering f.

We can therefore assume $g_{Y_1} = 0$ and without loss of generality $u \leq v$. Then (9.5) gives $v \leq 3 - \xi_h/2$. By Lemma 2.4, since h_1 is indecomposable, the greatest common divisor of all entries in $E_{h_1}(P_3)$ and $E_{h_2}(P_3) = E_{h_1}(P_4)$ is 1. Since in this case $\xi_h > 0$, (9.5) gives $v \leq 2$. If $u = v = 1$, $E_{h_1}(P_3) = [\ell]$ and $E_{h_1}(P_4) = E_{h_2}(P_3) = [\ell]$, contradicting the indecomposability of h_1 by Lemma 2.4.

It remains to consider the case $v = 2$. First let $u = 1$, so that (9.4) implies that the sum $\sum_{P\in Y(\mathbb{K})\setminus\{P_3,P_4\}} R_{h_1}(P)$ is 1. Hence h_1 has only one branch point Q other than P_3, P_4, and $E_{h_1}(Q) = [2, 1^{\ell-2}]$. Write $E_{h_2}(P_3) = [a, \ell - a]$. Since $E_{h_1}(P_3) = [\ell]$ and h_1 is indecomposable, Lemma 2.4 implies that $(a, \ell) = 1$. Hence the ramification type of h_1 over P_3, P_4, Q is type I1.1 $[\ell], [a, \ell - a], [2, 1^{\ell-2}]$ for some $1 \leq a < \ell$ with $(a, \ell) = 1$. By Remark 2.16, the corresponding ramification types for f in Table 3.1 are I1A.7a-I1A.7b.

It remains to consider the case $u = v = 2$ in which case $\sum_{P\in Y(\mathbb{K})\setminus\{P_3,P_4\}} R_{h_1}(P) = 2$ by (9.4). Write $E_{h_1}(P_3) = [a_1, a_2]$ and $E_{h_2}(P_3) = [b_1, b_2]$. By (9.2) and Lemma 9.2

$$4g_X = -2\ell + \sum_{1\le i,j\le 2} (a_i + b_j - 2(a_i, b_j)) + O_\alpha(1).$$

If $\{a_1, a_2\} \cap \{b_1, b_2\} = \emptyset$, then $\sum_{1\le i,j\le 2}(a_i + b_j - 2(a_i, b_j)) \ge 5\ell/2 - 5$ by Lemma 9.3, and hence $g_X > \ell/8 + O_\alpha(1)$. Thus, by choosing $c_2 < 1/8$ and sufficiently large d_2, the condition $g_X < c_2\ell - d_2$ forces $a_1 = b_1$ or b_2. Putting $a := a_1$, we deduce that $E_{h_1}(P_3) = E_{h_2}(P_3) = [a, \ell - a]$, and hence $E_{h_1}(P_4) = E_{h_2}(P_3) = [a, \ell - a]$. Since h_1 is indecomposable, Lemma 2.4 implies that $(a, \ell) = 1$.

Since $\sum_{P\in Y(\mathbb{K})\setminus\{P_3,P_4\}} R_{h_1}(P) = 2$, the ramification type of h_1 is one of the types I1A.4-6 in Table 11.1. By Remark 2.16, the corresponding ramification types for f are types I1A.4-I1A.6 in Table 3.1[4]

Case I1B: Estimating (9.2) using Lemma 9.2 gives:

$$4(g_X - 1) = 2\ell(g_{Y_1} - 1) + \ell \sum_{P\in Y(\mathbb{K})\setminus R} R_{h_1}(P) + \sum_{r_1,r_2\in E_{h_1}(P_i), i=1,2} (r_1 - (r_1, r_2)) + O_\alpha(1). \tag{9.6}$$

By Lemma 9.5, there exist $c_3 > 0$ and sufficiently large d_3 for which the conditions $g_X < c_3\ell - d_3$ and (9.6) force the ramification type of h_1 to be $[\ell], [a, \ell - a], [2, 1^{\ell-2}]$ with $(a, \ell) = 1$. Since the ramification types $[\ell]$ and $[a, \ell - a]$ appear over P_1 and P_2, Remark 2.16 implies that the corresponding ramification type for f is type I1.1 in Table 3.1. Choosing $c_2 \le c_3$ and $d_2 \ge d_3$, the conclusion follows when $g_X < c_2\ell - d_2$.

Case I2: Assume without loss of generality $m_{h_1}(P_1) = \infty$ and $m_{h_1}(P_2) = 1$. Let $P_3 \in S$ be a point with $m_{h_1}(P_3) = 2$, and set $P_4 = P_3^{\bar\sigma}$ so that $m_{h_1}(P_4) = 2$. Estimating (9.2) using Lemma 9.2 gives:

$$\begin{aligned} 4g_X =& 2\ell(g_{Y_1} - 1) + \sum_{r_1,r_2\in E_{h_1}(P_1)} (r_1 - (r_1, r_2)) + \ell \sum_{P\in Y(\mathbb{K})\setminus\{P_3,P_4\}} R_{h_1}(P) \\ &+ \ell \sum_{j=3}^{4} \Big(\frac{\ell}{2} - |E_{h_1}(P_j)| + |\{\text{odd } r \in E_{h_1}(P_j)\}|\Big) - \ell + O_\alpha(1). \end{aligned} \tag{9.7}$$

By Lemma 9.5, there exist $c_3 > 0$ and sufficiently large d_3 such that the conditions $g_X < c_3\ell - d_3$ and (9.7) force the ramification of h_1 to be one of the types I2.1–I2.15 in [26, Table 4.1]. As $m_{h_1}(P_1) = \infty$, and $m_{h_1}(P_2) = 1$, Remark 2.16 implies that the corresponding ramification types for f are types I2.1a–I2.15 in Table 3.1. Choosing $c_2 \le c_3$ and $d_2 \ge d_3$, the conclusion follows when $g_X < c_2\ell - d_2$.

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[4]The ramification data $([a, \ell - a], [a, \ell - a]), ([3, 1^{\ell-3}], 1)s, s$ and $([a, \ell - a], [a, \ell - a]), ([2^2, 1^{\ell-4}], 1)s, s$ are ruled out here as they do not correspond to a covering. Indeed, if there was a covering $f : X \to \mathbb{P}^1$ with such ramification then one would have $g_X < 0$ by Lemma 9.1.

Case F1A: Let $Q_1, Q_2 \in S$ be points with $m_{h_1}(Q_i) = 2$, for $i = 1, 2$ and set $Q_3 := Q_1^{\overline{\sigma}}$, and $Q_4 := Q_2^{\overline{\sigma}}$, so that $m_{h_1}(Q_i) = 2$, for $i = 3, 4$. Estimating (9.2) using Lemma 9.2 gives:

$$\begin{aligned} 4g_X =& 2\ell(g_{Y_1} - 1) + \ell \sum_{i=1}^{4} \big(\frac{\ell}{2} - |E_{h_1}(Q_i)| + |\{\text{odd } r \in E_{h_1}(Q_i)\}|\big) \\ &+ \ell \sum_{P \in Y(\mathbb{K}) \setminus \{Q_1,\dots,Q_4\}} R_{h_1}(P) - 2\ell + O_\alpha(1). \end{aligned} \tag{9.8}$$

By Lemma 9.5, there exist $c_3 > 0$ and sufficiently large d_3 such that the condition $g_X < c_3\ell - d_3$ and (9.8) force the ramification type of h_1 to be one of the types (F1.1)-(F1.9) in [26, Table 4.1]. As $m_{h_1}(P_1) = m_{h_1}(P_2) = 1$, Remark 2.16 implies that the corresponding ramification types for f are types F1A.1a-F1A.9 in Table 3.1. Choosing $c_2 \le c_3$ and $d_2 \ge d_3$, the conclusion follows when $g_X < c_2\ell - d_2$.

Case F1B: Let $P_3 \in S$ be a point with $m_{h_1}(P_3) = 2$, and set $P_4 = P_3^{\overline{\sigma}}$, so that $m_{h_1}(P_i) = 2$ for $i = 1, \dots, 4$. Then (9.2) gives:

$$\begin{aligned} 4g_X =& 2\ell(g_{Y_1} - 1) + \ell \sum_{i=1}^{4} \big(\frac{\ell}{2} - |E_{h_1}(P_i)| + |\{\text{odd } r \in E_{h_1}(P_i)\}|\big) \\ &+ \ell \sum_{P \in Y(\mathbb{K}) \setminus \{P_1,\dots,P_4\}} R_{h_1}(P_i) + O_\alpha(1). \end{aligned} \tag{9.9}$$

On the other hand by the Riemann-Hurwitz formula for h_1, we have

$$2(g_{Y_1} - 1) = \sum_{P \in Y(\mathbb{K}) \setminus \{P_1,\dots,P_4\}} R_{h_1}(P_i) + \sum_{i=1}^{4} (\frac{\ell}{2} - |E_{h_1}(P_i)|). \tag{9.10}$$

Combining the latter equality with (9.9) one has

$$4g_X = 4\ell(g_{Y_1} - 1) + \ell \sum_{i=1}^{4} |\{\text{odd } r \in E_{h_1}(P_i)\}|. \tag{9.11}$$

Hence for $c_2 < 1/4$ and sufficiently large d_2, the condition $g_X < c_2\ell - d_2$ forces:

$$0 = 4(g_{Y_1} - 1) + \sum_{i=1}^{4} |\{\text{odd } r \in E_{h_1}(P_i)\}|. \tag{9.12}$$

In particular, $g_{Y_1} \le 1$. Furthermore, if $g_{Y_1} = 1$, then (9.12) and (9.10) imply that the ramification type of h_1 is $[2^{\ell/2}]$ four times. In this case Lemma 4.7 shows that $\mathrm{Mon}(h_1)$ is solvable, contradicting the nonsolvability of G. If $g_{Y_1} = 0$, then (9.12) and (9.10) imply that the ramification type of h_1 is $[1, 2^{(\ell-1)/2}]$ four times. In this case as well, $\mathrm{Mon}(h_1)$ is solvable by Lemma 4.7, contradicting the nonsolvability of G.

Case F2: Assume without loss of generality $m_{h_1}(P_1) = 3$, and $m_{h_1}(P_2) = 1$. Let $P_3 \in S$ be a point with $m_{h_1}(P_3) = 3$ and set $P_4 := P_3^{\overline{\sigma}}$, so that $m_{h_1}(P_4) = 3$. Estimating (9.2) using Lemma 9.2 gives:

$$(9.13)\quad \begin{aligned} 4g_X &= 2\ell(g_{Y_1}-1) + \ell \sum_{i\in\{1,3,4\}} \big(\frac{\ell}{3} - |E_{h_1}(P_i)| + \frac{4}{3}|\{r \in E_{h_1}(P_i) \,|\, (r,3)=1\}|\big) \\ &+ \ell \sum_{P\in Y(\mathbb{K})\setminus\{P_1,P_3,P_4\}} R_{h_1}(P) - 4\ell/3 + O_\alpha(1). \end{aligned}$$

By the Riemann–Hurwitz formula for h_1, we have

$$(9.14)\quad 2(g_{Y_1}-1) = \sum_{i\in\{1,3,4\}} \big(\frac{\ell}{3} - |E_{h_1}(P_i)|\big) + \sum_{P\in Y(\mathbb{K})\setminus\{P_1,P_3,P_4\}} R_{h_1}(P).$$

Combining the latter with (9.13), one has

$$4g_X = 4\ell(g_{Y_1}-1) + \frac{4}{3}\ell \sum_{i\in\{1,3,4\}} |\{r \in E_{h_1}(P_i) \,|\, (r,3)=1\}| - \frac{4}{3}\ell.$$

Hence for $c_2 < 1/12$ and sufficiently large d_2, the condition $g_X < c_2\ell - d_2$ forces:

$$0 = 4(g_{Y_1}-1) + \frac{4}{3} \sum_{i\in\{1,3,4\}} |\{r \in E_{h_1}(P_i) \,|\, (r,3)=1\}| - \frac{4}{3},$$

or equivalently,

$$(9.15)\quad 4 - 3g_{Y_1} = \sum_{i\in\{1,3,4\}} |\{r \in E_{h_1}(P_i) \,|\, (r,3)=1\}|.$$

In particular $g_{Y_1} \le 1$. There are no ramification types for h_1 satisfying (9.15) and (9.14) with $g_{Y_1} = 1$. If $g_{Y_1} = 0$, at least two of $E_{h_1}(P_1), E_{h_1}(P_3), E_{h_1}(P_4)$ contain an entry r prime to 3, by Lemma 2.4. The ramification types for h_1 satisfying the latter constraint, (9.15) and (9.14) are types F2.1-F2.3 in Table 11.1. As $m_{h_1}(P_1) = 3$ and $m_{h_1}(P_2) = 1$, Remark 2.16 implies that the corresponding ramification types for f are types F2.1-F2.3 in Table 3.1.

Case F3: Assume without loss of generality $m_{h_1}(P_1) = 2$ and $m_{h_1}(P_2) = 1$. Let $P_3 \in S$ be a point with $m_{h_1}(P_3) = 4$, and set $P_4 = P_3^{\overline{\sigma}}$, so that $m_{h_1}(P_4) = 4$. Estimating (9.2) using Lemma 9.2 gives:

$$(9.16)\quad \begin{aligned} 4g_X &= 2\ell(g_{Y_1}-1) + \ell\Big(\frac{\ell}{2} - |E_{h_1}(P_1)| + |\{\text{odd } r \in E_{h_1}(P_1)\}|\Big) \\ &+ \ell\sum_{i=3}^{4}\Big(\frac{\ell}{4} - |E_{h_1}(P_i)| + |\{r \in E_{h_1}(P_i) \,|\, r \equiv 2\,(4)\}| + \frac{3}{2}|\{\text{odd } r \in E_{h_1}(P_i)\}|\Big) \\ &+ \ell \sum_{P\in Y(\mathbb{K})\setminus\{P_1,P_3,P_4\}} R_{h_1}(P) - \ell + O_\alpha(1). \end{aligned}$$

By Lemma 9.5, there exists $c_3 > 0$ and sufficiently large d_3 such that the conditions $g_X < c_3\ell - d_3$ and (9.16) force the ramification type of h_1 to be one of types F3.1-F3.3 in [26, Table 4.1]. As $m_{h_1}(P_1) = 2$ and $m_{h_1}(P_2) = 1$, Remark 2.16 implies that the corresponding ramification types for f are types F3.1-F3.3 in Table 3.1. Choosing $c_2 \leq c_3$ and $d_2 \geq d_3$, the conclusion follows when $g_X < c_2\ell - d_2$.

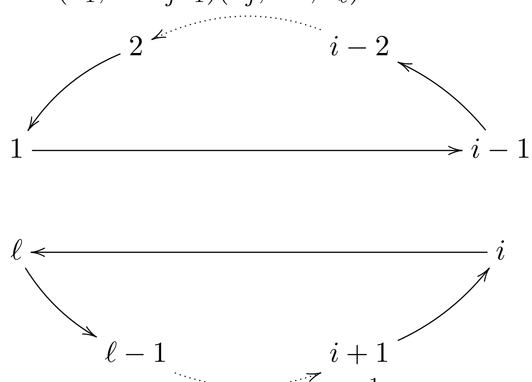
10. Non-occuring ramification types

It remains to verify that the ramification types in Lemma 9.6 do not correspond to an indecomposable covering. The proof uses the following lemmas:

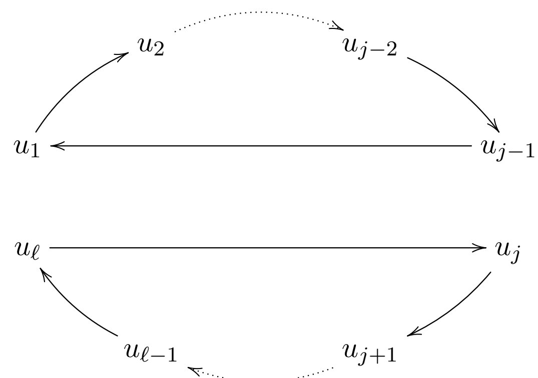
Lemma 10.1. *Let $a, b \in S_\ell$ be ℓ-cycles and $c, d \in S_\ell$ be 2-cycles.*

(1) *If $adbc = 1$, then there exists $z \in S_\ell$ such that $z^2 = 1$, $zaz = b$ and $zcz = d$.*
(2) *If c, d are disjoint 2-cycles, and $abcd = 1$, then there exists $v \in S_\ell$ such that $v^2 = cd$ and $vbv^{-1} = a$ and $vav^{-1} = cdbcd$.*

Proof. First suppose that $adbc = 1$ or $ad = cb^{-1}$. Without loss of generality, assume $b = (1, \ldots \ell)$ and $c = (1, i)$. Then cb^{-1} is a product of the two disjoint cycles $(i - 1, \ldots, 1)$ and $(\ell, \ldots, i)$. Similarly, writing $a = (u_1, \ldots, u_\ell)$ and $d = (u_1, u_j)$ we get $ad = (u_1, \ldots u_{j-1})(u_j, \ldots, u_\ell)$.



The element cb^{-1}



The element ad

Since $ad = cb^{-1}$ we get either $i - 1 = j - 1$ or $i - 1 = \ell - j + 1$. Without loss of generality assume $i = j$.

Defining $z \in S_\ell$ by $z(i) = u_i$, we get $(cb^{-1})^z = (ad)^{-1}$ and $c^z = d$. Hence $(bc)^z = ad$ and $b^z = a$. Since $(bc)^z = ad = (bc)^{-1}$, we deduce that z^2 acts trivially by conjugation on bc and on c. Since b and c generate S_ℓ, z^2 is in the center of S_ℓ, so $z^2 = 1$, completing part (1).

Assume $abcd = 1$ and rewrite this relation as $a_0dbc = 1$ where $a_0 := a^{d^{-1}} = a^d$. Applying part (1) to the latter relation, we obtain an involution $z \in S_\ell$ such that $(a_0)^z = b$ and $c^z = d$. Setting $v = dz$, we have $a^v = (a_0^d)^v = a_0^z = b$, and $v^2 = dz^2d^z = dc = cd$, proving (2). □

The proof of Corollary 9.6 for type F4.N1 and F4.N2 is based on the following theorem. Let $[a, b] := a^{-1}b^{-1}ab$ denote the commutator.

Theorem 10.2. *Let a, b be elements of S_ℓ such that $[a, b]$ is a product of two transpositions and $\langle a, b\rangle$ contains A_ℓ. Then there exists $z \in S_\ell$ such that $a^z = a^{-1}$ and $b^z = b^{-1}$.*

Proof of Lemma 9.6. We show there is no product one tuple $x_1, \dots, x_r$ corresponding to the ramification types in Table 9.1 and generating a primitive group. In case I1A.N1 by conjugating the tuple by an element in S_ℓ^2, we may assume that the product 1 tuple is of the form:

$$(a, b), (c, d), (e^{-1}, e)s, s,$$

where a and b are ℓ-cycles, and c and d are 2-cycles. The product 1 relation gives $e = ac$ and $bde = bdac = 1$. By Lemma 10.1, there exists $z \in S_\ell$ such that $z^2 = 1$, $b = zaz$, $d = zcz$, so that

$$ze^{-1}z = zbdz = (zbz)(zdz) = ac = e,$$

$zbz = a$, $zdz = c$, and $zez = e^{-1}$. Since $K := G \cap S_\ell^2$ is generated by $(a, b), (c, d)$ and (e, e^{-1}), it is contained in $\{(u, zuz) : u \in S_\ell\}$, hence doesn't contain A_ℓ^2 and does not generate a primitive subgroup $G \le S_\ell \wr S_2$ by Lemma 2.13.

In case I1A.N2, conjugating the tuple with an element in S_ℓ^2, we may assume that the product 1 tuple is of the form:

$$(b, a), (cd, 1)s, (e, e^{-1})s$$

where c, d are disjoint 2-cycles and a, b are ℓ-cycles. The product 1 relation gives $e = bcd$, and $ae = abcd = 1$. By the previous lemma there exists $v \in S_\ell$ such that $v^2 = cd$ and $vbv^{-1} = a$ and $vav^{-1} = cdbcd$; since the intersection of the group with S_ℓ^2 is generated by $(b, a), (cd, cd)$ and $(a, cdbcd)$ (which is conjugate to the product of $(cd, 1)s$ and $(e, e^{-1})s$), it is contained in $\{(u, vuv^{-1}) \,|\, u \in S_\ell\}$, hence doesn't contain A_ℓ^2 and does not generate a primitive subgroup $G \le S_\ell \wr S_2$ by Lemma 2.13.

In cases F4.N1, F4.N2, conjugating by an element in S_ℓ^2, we may assume that the product one tuple is of the form:

$$s, (a^{-1}, a)s, (b, b^{-1})s, (c^{-1}v, c)s,$$

where v is a 3-cycle or the product of two disjoint 2-cycles. The product 1 relation amounts to the equalities $abc = 1$ and $a^{-1}b^{-1}c^{-1}v = 1$. Replacing $c = b^{-1}a^{-1}$, we have $[a, b] = a^{-1}b^{-1}ab = v^{-1}$. Since $(a^{-1}, a) = (a^{-1}, a)s \cdot s \in K$ and similarly $(b^{-1}, b) \in K$, the projections of K to the i-th coordinate is $K_i = \langle a, b\rangle$, for $i = 1, 2$. Since G is primitive, K_i is primitive by Remark 2.12.(3). Since K_i is primitive and contains v, which is either a 3-cycle or a product of two disjoint transpositions, Remark 3.2 implies that $K_i \supseteq A_\ell$, for $i = 1, 2$. Since $K_i \supseteq A_\ell$, $i = 1, 2$, and G is primitive, we have $G \supseteq A_\ell^2$ by Lemma 2.13. On the other hand, by Theorem 10.2, there exists $z \in S_\ell$ such that $a^z = a^{-1}$ and $b^z = b^{-1}$. Since K is generated by (a^{-1}, a) and (b^{-1}, b), we get that $K \subseteq \{(u, u^z) : u \in S_\ell\}$, contradicting $K \supseteq A_\ell^2$. □

11. Realizing the ramification types in Table 3.1

In this section, we prove the existence of indecomposable coverings $f : X \to \mathbb{P}^1$ with the ramification types in Table 3.1, and a monodromy group of product type. For this, by Riemann's existence theorem it suffices to show:

Proposition 11.1. *For every ramification type in Table 3.1 with $\ell \geq 9$ there exists a product* 1 *tuple corresponding to the ramification type and generating a primitive group $G \leq S_\ell \wr S_2$ that contains A_ℓ^2.*

The proof relies on the following lemma:

Lemma 11.2. *For each $\ell \geq 9$ in the appropriate congruence class modulo* 3 *and integer $0 < a < \ell/2$ with $(a,\ell) = 1$, there exist $x_1, x_2, x_3 \in S_\ell$ such that $x_1x_2x_3 = 1$, and $\langle x_1, x_2, x_3\rangle$ contains A_ℓ, and the cycle structures of x_1, x_2, x_3 are one of the following:*

(11.1)

$I1A.1$	$[\ell], [\ell], [3, 1^{\ell-3}]$
$I1A.2$	$[\ell], [\ell], [2, 1^{\ell-2}], [2, 1^{\ell-2}]$
$I1A.3$	$[\ell], [\ell], [2^2, 1^{\ell-4}]$
$I1A.4$	$[a, \ell-a], [a, \ell-a], [3, 1^{\ell-3}]$
$I1A.5$	$[a, \ell-a], [a, \ell-a], [2, 1^{\ell-2}], [2, 1^{\ell-2}]$
$I1A.6$	$[a, \ell-a], [a, \ell-a], [2^2, 1^{\ell-4}]$
$F2.1$	$[3^{\ell/3}], [1, 2, 3^{(\ell-3)/3}]$ *twice*
$F2.2$	$[1^2, 3^{(\ell-2)/3}], [2, 3^{(\ell-2)/3}]$ *twice*
$F2.3$	$[2^2, 3^{(\ell-4)/3}], [1, 3^{(\ell-1)/3}]$ *twice*

Proof of Proposition 11.1. In each case in Table 3.1, we give a product 1 tuple corresponding to this ramification types. Let $G \leq S_\ell \wr S_2$ be the group generated by this tuples, $K := G \cap S_\ell^2$ and $\pi_i : K \to S_\ell, i = 1, 2$ be the natural projections to the first and second coordinate, respectively.

Consider elements $a, b, c, d, f \in S_\ell$ satisfying $dcabf = 1$ with $\langle a, b, c, d\rangle = A_\ell$ or S_ℓ and such that a, d are nonconjugate in S_ℓ. Then both tuples

$$(a,d), (b,c), (f,1), s, (abf, (abf)^{-1})s; \text{ and} \\ (a,d), (b,c), (f,1)s, (abf, (abf)^{-1})s, \tag{11.2}$$

have product 1. Furthermore, we note that G is primitive: Indeed, G contains $(a, d), (b, c)$ and the conjugates of these elements by $(f, 1)s$, which are $(d, f^{-1}af)$ and $(c, f^{-1}bf)$, respectively. Then the image $\pi_1(K)$ contains $J := \langle a, b, c, d\rangle$ and hence contains A_ℓ. Also G contains the square of $(f, 1)s$ which is (f, f), so that $\pi_2(K)$ contains $f, d, c, f^{-1}af, f^{-1}bf$ and hence contains J, and also A_ℓ. Thus, we can apply Lemma 2.13, and as a, d are nonconjugate deduce that $K \supseteq A_\ell^2$ and G is primitive.

Recall that an element of the form $(\alpha, \alpha^{-1})s$, $\alpha \in S_\ell^2$, is conjugate in $S_\ell \wr S_2$ to s. Thus each of the ramification types in Table 3.1 corresponds to one of the product 1 tuples in (11.2), with the exception of cases

(11.3) I1.1, I1A.2a, I1A.5a, I2.1b, I2.2a, I2.9a, I2.10b,

where none of the branch cycles are conjugate to s, and all F4 cases in which there are four elements in $G \setminus K$. Moreover, for each ramification type which is not in (11.3), there exists a, b, c, d, f for which (11.2) corresponds to this ramification type, and such that $dcabf = 1$ and $\langle a, b, c, d\rangle$ is A_ℓ or S_ℓ by [26, Section 11] which relies on [16, Chapter 3] and by Lemma 11.2 for the I1A and F2 types, as desired.

We next consider the exceptions in (11.3). Consider the tuple

$$(b,d),(1,u)s,(bv,b^{-1})s, \tag{11.4}$$

where $b,d,u,v \in S_\ell$ are elements which satisfy $dubv = 1$ and $\langle b,d,u,v\rangle = A_\ell$ or S_ℓ. The tuple (11.4) then has product 1. We claim that $\pi_i(K)$ contains A_ℓ, for $i = 1,2$. Since K contains (b,d) and the squares $((1,u)s)^2 = (u,u)$ and $((bv,b^{-1})s)^2 = (bvb^{-1},v)$, the projection $\pi_2(K)$ contains d,u,v, and hence also $b = u^{-1}d^{-1}v^{-1}$, so that $\pi_2(K) \supseteq A_\ell$. Conjugating $\pi_2^{-1}(A_\ell)$ by $(1,u)s$ we get that $\pi_1(K)$ also contains A_ℓ, proving the claim. Hence by Lemma 2.13, either $G \supseteq A_\ell^2$ or $K \subseteq \{(u,u^w)\,|\,u \in S_\ell\}$ for some $w \in S_\ell$.

For each of the ramification types $\mathcal{R}$ in (11.3), there exist $d,u,b,v \in S_\ell$ such that $dubv = 1$, $\langle b,d,u,v\rangle = A_\ell$ or S_ℓ, and for which the tuple (11.4) is conjugate to $\mathcal{R}$, by [26, Section 11] and by Lemma 11.2 for the I1A types. In cases I1.1, I2.1b, I2.9a, the elements a,d can be chosen to be nonconjugate, and hence $K \supseteq A_\ell^2$ and G is primitive by Lemma 2.13.

Other than the F4, types, it remains to choose b,d,u,v in cases I2.2a, I2.10b, I1A.2a, I1A.5a such that $K \supseteq A_\ell^2$ or equivalently that K is not contained $D_w := \{(u,u^w)\,|\,u \in S_\ell\}$ for any $w \in S_\ell$. In case I2.2a, we may choose

$$b = (1,2)(3,4)\cdots(\ell-3,\ell-2), \qquad v = (\ell-1,\ell)$$
$$d = (2,3)(4,5)\cdots(\ell-2,\ell-1), \qquad u = (2,4,\dots,\ell,\ell-1,\ell-3,\dots,1).$$

As K contains $(u,u) = ((1,u)s)^2$ and (b,d), an inclusion $K \subseteq D_w$ implies that $u^w = u$ and $b^w = d$. As the centralizer of the ℓ-cycle u is $\langle u\rangle$, this implies $w \in \langle u\rangle$, but a straightforward check shows that $b^{u^i} \neq d$ for all i, contradiction. Hence $K \supseteq A_\ell^2$ and G is primitive by Lemma 2.13.

The same argument also applies in case I2.10b by choosing:

$$b = (1,2)(3,4)\cdots(\ell-2,\ell-1), \quad v = (2k,1), \quad d = (2,3)(4,5)\cdots(\ell-1,\ell)$$
$$u = (2k-1,2k-3,\dots,1)(2,4,\dots,\ell-1,\ell,\ell-2,\dots,2k+1),$$

for every $1 \le k < \ell-1$ prime to ℓ. As in the previous case, an inclusion $K \subseteq D_w$ implies $u^w = u$ and $b^w = d$. As $(k,\ell) = 1$, the centralizer of u is

$$C_{S_\ell}(u) = \langle (1,3,\dots,2k-1),(2k+1,2k+3,\dots,\ell,\ell-1,\ell-3,\dots,2)\rangle.$$

Since $b^w = d$, we have $w(\ell) = 1$ but there is no such element in $C_{S_\ell}(u)$.

In Case I1A.2a (resp. I1A.5a), we can take $b = d^{-1}$ to be $(1,\dots,\ell)$ (resp. to be $(1,\dots,k)(k+1,\dots,\ell)$), and $u^b = v = (2,3)$ (resp. $u^b = v = (1,k+1)$). Conjugating (b,b^{-1}) by $(1,u)s$ we get $((b^{-1})^u,b) \in G$, and hence $(u^{-1}b^{-1}ub,1) \in G$. Since $u^{-1}b^{-1}ub$ is not conjugate to 1 in both cases, Lemma 2.13 gives $K \supseteq A_\ell^2$ and hence G is primitive.

For types F4.1-F4.6, consider the tuple:

$$s,(a^{-1},a)s,(b,ub^{-1})s,(c^{-1}v,c)s,(d_1,d_2),(e_1,e_2), \tag{11.5}$$

where all of $u,v,d_i,e_i, i = 1,2$, are trivial except (A) one 3-cycle in case F4.6, (B) one element which is a product of two transpositions in case F4.5, and (C) two elements that are transpositions in cases F4.1-F4.3. The tuple in case F4.4 is specified in the end of the proof. Note that the tuples in (11.5) cover all cases F4.1-F4.6 as they are conjugate in $S_\ell \wr S_2$ to:

$$s, s, (u,1)s, (v,1)s, (d_1,d_2), (e_1,e_2). \tag{11.6}$$

We first consider cases F4.1-F4.3, F4.5, and F4.6. The product 1 relation for the elements in (11.5) is equivalent to the equations $abcd_1e_1 = 1$ and $a^{-1}ub^{-1}c^{-1}vd_2e_2 = 1$. Setting $a^{-1} = bcd_1e_1$, we get that the product 1 relation is equivalent to:

$$[c^{-1}, b^{-1}] = (d_1e_1u)^{b^{-1}c^{-1}}vd_2e_2,$$

where $[c^{-1}, b^{-1}] = cbc^{-1}b^{-1}$. Setting $b = (1,\dots,\ell)$ and $c = (1,k)$ with $k > 2$ and $(k-1,\ell) = 1$ (resp. $c = (1,2)$, $k = 2$) in cases (B) and (C) (resp. in case (A)), we get that $[c^{-1}, b^{-1}]$ is a product of two transpositions (resp. a 3-cycle). For each of the cases F4.5-F4.6, we then choose from $u, v, d_i, e_i, i = 1,2$ the corresponding nontrivial element in cases (B) and (C) (resp. two elements in case (A)) so that the tuple (11.5) has product 1.

We next claim that the group G generated by the tuple (11.5) is primitive in cases F4.1-F4.3, and F4.5-F4.6. In these cases, the projections $\pi_i(K)$, $i = 1,2$ are all of S_ℓ since these contain the ℓ-cycle b and the transposition $(1,k)$ (as $(k-1,\ell) = 1$). Note that in cases F4.2, F4.3, F4.5 and F4.6, the tuple (11.5) contains an element of the form $(x,1)$ with $x \neq 1$, arising either from (d_1,d_2) or from (e_1,e_2), and thus K is not contained in D_w for any $w \in S_\ell$, implying that G is primitive by Lemma 2.13. In case F4.1 we can assume v is a transposition, and hence $(c^{-1}v, c) \in K$, but c and $c^{-1}v$ are nonconjugate in S_ℓ (as they have a different sign), hence $K \not\subseteq D_w$ for all $w \in S_\ell$, and G is primitive by Lemma 2.13.

It remains to consider the case F4.4. In this case, we use the tuple (11.5) with $u = v = e_1 = e_2 = 1$. Letting $3 \le k < \ell$ be an integer prime to ℓ, we set $a := (1,2,\dots,k)(k+1,\dots,\ell)$, $c := (1,2,k+1,3)$, and $d_1 := (1,k+1)$. We then let $b = a^{-1}d_1c^{-1}$, and $d_2 = d_1^{ac^{-1}}$, so that $abcd_1 = 1$ and $a^{-1}b^{-1}c^{-1}d_2 = 1$, and hence the tuple (11.5) satisfies the product 1 relation. We next show that the group G generated by this tuple is primitive. Since both projections $\pi_i(K)$, $i = 1,2$ of $K = G \cap S_\ell^2$ contain a and d_1, it also contains $\langle a, d_1\rangle$ which is all of S_ℓ as $(k,\ell) = 1$. As in addition (a, a^{-1}) and (c, c^{-1}) are in K, in order to show that G is primitive by Lemma 2.13, we claim that there is no element $w \in S_\ell$ such that $a^w = a^{-1}$ and $c^w = c^{-1}$. Such $w \in S_\ell$ would preserve the sets $\{1,\dots,k\}, \{1,2,k+1,3\}$, and hence also $\{1,2,3\}$ and $\{k+1\}$. An element w which satisfies $a^w = a^{-1}$ and preserves $\{1,2,3\}$, must fix 2, and hence satisfies $c^w = (3,2,k+1,1) \neq c$, proving the claim. The primitivity of G then follows from Lemma 2.13. □

Proof of Lemma 11.2. In each case let G be the group generated by the elements in the corresponding product 1 tuple.

Cases I1A.1,I1A.2,I1A.3: Write the product 1 relation as $x_2x_3 = x_1^{-1}$ where

$$x_2 := (1,u+1)(2,u+2), \qquad x_3 := (1,2,\dots,\ell),$$
$$x_1^{-1} := (1,u+2,3,4,\dots,u+1,2,u+3,u+4,\dots,\ell),$$

where $1 \le u \le \ell - 2$. For $u > 1$, this covers the product 1 relation in case I1A.3 and also in the case IA.2 by splitting x_2 into two elements $(1, u+1)$ and $(2, u+2)$. Note that if $u = 1$ then $(1, u+1)(2, u+2) = (3, 2, 1)$ is a 3-cycle, covering case I1A.1.

Now suppose that $(u, \ell) = 1$ (which is of course true for $u = 1$). Let $G = \langle x_1^{-1}, x_2 \rangle$. Consider a partition of $\{1, 2, \ldots, \ell\}$ preserved by G, and suppose that some part B has size at least 2. Since the partition is preserved by $\langle x_3 \rangle$, it must consist of congruence classes mod some proper divisor of ℓ. Suppose that $1 \in B$ and B is not $\{1, 2, \ldots, \ell\}$. Then $2 \notin B$ and $u + 1 \notin B$, as $(u, \ell) = 1$. Hence B is not fixed by $(1, u+1)(2, u+2)$, so it contains no fixed points of this permutation, whence $B = \{1, u+2\}$, so that $u = \ell/2 - 1$; but then its image $\{2, u+1\} = \{2, \ell/2\}$ is also a block, contradicting the fact that the block containing $\ell/2$ is $\{\ell/2, \ell\}$. Therefore the partition is trivial and G is primitive. Since G contains an element with cycle structure $[2^2, 1^{\ell-4}]$ or $[3, 1^{\ell-3}]$, it follows that G contains A_ℓ, by Remark 3.2.

Case I1A.4: Write the product 1 relation as $x_2 x_3 = x_1^{-1}$ where

$$x_2 := (1, a+1, \ell+1-a), \qquad x_3 := (1, 2, \ldots, a)(a+1, a+2, \ldots, \ell),$$
$$x_1^{-1} := (1, a+2, a+3, \ldots, \ell+1-a, 2, 3, \ldots, a)(a+1, \ell+2-a, \ell+3-a, \ldots, \ell).$$

Any block B of G of size bigger than 1 which contains 1 must also contain a fixed point of x_2 unless $B = \{1, a+1, \ell+1-a\}$, so in any case B contains $\{1, a+1, \ell+1-a\}$. But the a-th power of x_3 is an $(\ell - a)$-cycle which fixes 1 and hence B. As B contains $a + 1$, it also contains the entire $(\ell - a)$-cycle. Likewise the $(\ell - a)$-th power of x_3 is an a-cycle which contains 1 and fixes $a + 1$, so it fixes B and hence contains the entire a-cycle. So $B = \{1, 2, \ldots, \ell\}$, whence G is primitive. Since G contains a 3-cycle, it contains A_ℓ by Remark 3.2.

Case I1A.5: Use the identity $xx^{-1}yy^{-1} = 1$, where

$$x := (1, 2, \ldots, a)(a+1, a+2, \ldots, \ell), \text{ and } y := (1, a+1),$$

for $(a, \ell) = 1$. As in case I1A.4, $\langle x, y \rangle$ is primitive and hence is S_ℓ by Remark 3.2.

Case I1A.6: Write the product 1 relation as $x_1 x_2 = x_3$, with

$$x_1 := (a+1, a, a-1, \ldots, 2)(\ell, \ell-1, ..., a+2, 1),$$
$$x_2 := (1, 2, \ldots, a)(a+1, \ldots, \ell), \qquad x_3 := (1, a+1)(2, a+2)$$

if $1 < a < \ell$, and

$$x_1 := (\ell-1, ..., 3, \ell, 2), \quad x_2 := (1, 2, ..., \ell-1), \quad x_3 := (1, 2)(3, \ell)$$

if $a = 1$ or ℓ. If $1 < a < \ell$, switching the roles of x_2, x_3, the same argument as in case I1A.4 applies unless the nontrivial block B containing 1 contains no element from $a+1, \ldots, \ell$ and no fixed point of x_3, so that $B = \{1, 2\}$. In this case B is not fixed by x_2, contradicting $1^{x_2} = 2$. If $a = 1$ or ℓ, and C is a block containing ℓ, then C is fixed by x_2, and hence either $C = \{\ell\}$ or C also contains the orbit $1, 2, \ldots, \ell - 1$ of x_2. Thus G is primitive. As in addition G contains x_3, it contains A_ℓ by Remark 3.2.

Case F3.2: For $\ell = 3u+1$, write the product 1 relation as $x_1x_2 = x_3^{-1}$ where

$$x_1 := (3u)(1,2,3u+1)\prod_{i=1}^{u-1}(3i,3i+1,3i+2),$$

$$x_2 := (3u+1)\prod_{i=0}^{u-1}(3i+1,3i+2,3i+3),$$

$$x_3^{-1} := (2,3u+1)(3u-2,3u)\prod_{i=1}^{u-1}(3i-2,3i,3i+2).$$

Consider a block B which contains 2 and has size at least 2. If $3u+1 \notin B$ then B is not fixed by $(x_3^{-1})^3 = (2,3u+1)(3u-2,3u)$, so B is either $\{2,3u-2\}$ or $\{2,3u\}$. But $3u \notin B$, since $3u$ is fixed by x_1, but $2^{x_1} = 3u+1$. So $B = \{2,3u-2\}$. Then $B^{x_3^{-1}} = \{3u+1,3u\}$, while $B^{x_1} = \{3u+1,3u-1\}$, contradiction. So B has to contain $3u+1$, and thus is fixed by x_1 and x_3^{-1}, whence the block contains the orbit of 2 under G, so $B = \{1,2,\dots,\ell\}$. Hence G is primitive and contains an element of cycle structure $[2^2,1^{\ell-4}]$, so $G = A_\ell$ by Remark 3.2.

Case F2.2: For $\ell = 3u+2$, write the product 1 relation as $x_1x_2 = x_3^{-1}$ where

$$x_1 := (2,3u+2)(1,3,4)\prod_{i=2}^{u}(3i-1,3i,3i+1), \quad x_2 := (1)(2)\prod_{i=1}^{u}(3i,3i+1,3i+2),$$

$$x_3 := (1,4)(2,3u,3u+2)\prod_{i=1}^{u-1}(3i,3i+2,3i+4).$$

Consider a block B which contains 2 and has size at least 2. Since $x_1^3 = (2,3u+2)$, the block contains $3u+2$ and hence is fixed by x_3^{-1}. Thus B contains the orbit of $3u+2$ under $G = \langle x_1, x_3\rangle$, which is $\{1,2,\dots,\ell\}$. So G is primitive and contains a 2-cycle, whence $G = S_\ell$ by Remark 3.2.

Case F2.1: For $\ell = 3u$, write the product 1 relation as $x_1x_2 = x_3^{-1}$ where

$$x_1 := (1,2,3u)(3u-2,3u-1)\prod_{i=1}^{u-2}(3i,3i+1,3i+2), \quad x_2 := \prod_{i=0}^{u-1}(3i+1,3i+2,3i+3),$$

$$x_3^{-1} := (3u-1)(3u-3,3u-5)(2,3u-2,3u)\prod_{i=1}^{u-2}(3i-2,3i,3i+2).$$

The same argument as in case F2.2 applied to the block B of $3u-1$ gives $G = S_\ell$. □

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