RATIONAL PULLBACKS OF GALOIS COVERS

PIERRE DÈBES, JOACHIM KÖNIG, FRANÇOIS LEGRAND, AND DANNY NEFTIN

Abstract. The finite subgroups of $\mathrm{PGL}_2(\mathbb{C})$ are shown to be the only finite groups G with this property: for some integer r_0 (depending on G), all Galois covers $X \to \mathbb{P}^1_{\mathbb{C}}$ of group G can be obtained by pulling back those with at most r_0 branch points along non-constant rational maps $\mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$. For $G \subset \mathrm{PGL}_2(\mathbb{C})$, it is in fact enough to pull back one well-chosen cover with at most 3 branch points. A consequence of the converse for inverse Galois theory is that, for $G \not\subset \mathrm{PGL}_2(\mathbb{C})$, letting the branch point number grow provides truly new Galois realizations $F/\mathbb{C}(T)$ of G. Another application is that the "Beckmann–Black" property that "any two Galois covers of $\mathbb{P}^1_{\mathbb{C}}$ with the same group G are always pullbacks of another Galois cover of group G" only holds if $G \subset \mathrm{PGL}_2(\mathbb{C})$.

1. Introduction

Suppose $f : X \to \mathbb{P}^1_k$ is a *k-regular cover*, i.e., a (branched) cover over a field k with X a smooth and geometrically irreducible curve over k. By a *rational pullback* of f, we mean a k-regular cover $f_{T_0} : X_{T_0} \to \mathbb{P}^1_k$ obtained by pulling back f along some non-constant rational map $T_0 : \mathbb{P}^1_k \to \mathbb{P}^1_k$. If f is given by a polynomial equation $P(t,y) = 0$ (with f corresponding to the t-coordinate projection) and T_0 is viewed as a rational function $T_0(U) \in k(U)$, then an equation for the pullback f_{T_0} is merely $P(T_0(u), y) = 0$. As recalled in §2.2, if f is additionally Galois of group G, then for "many" T_0, the resulting map f_{T_0} remains a k-regular Galois cover of group G.

The *Regular Inverse Galois Problem* over k precisely consists in realizing each finite group as the Galois group of a k-regular Galois cover of $\mathbb{P}^1_k$. Rational pullback creates such covers, if one is already known. Finding covers g that are *not* rational pullbacks of some Galois covers f of given group G may be a more important issue. Indeed, if none of the f are defined over k (as a regular cover), the pullbacks f_{T_0} are generally not expected to be either, in which case the remaining hope to realize G as a regular Galois group over k rests on those covers g.

1.1. **Main results.** Assume first $k = \mathbb{C}$. Consider the situation that all Galois covers $X \to \mathbb{P}^1_{\mathbb{C}}$ of given group G can be obtained from a proper subset of them by rational pullback; say then that the subset is *regularly parametric*[1]. For some finite groups G, a single cover f may suffice. For example, the degree 2 cover $\mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ sending z to z^2 *is* regularly parametric. Such situations are, however, exceptional. For "general" finite groups, an opposite conclusion holds:

Theorem 1.1. *The finite subgroups of* $\mathrm{PGL}_2(\mathbb{C})$ *(i.e., cyclic and dihedral groups, A_4, S_4, and A_5) are exactly those finite groups which have a regularly parametric cover $X \to \mathbb{P}^1_{\mathbb{C}}$. More precisely, given a finite group G, the following two statements hold:*

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[1]The term "regularly" will be fully justified with the general definition of "k-regular parametricity" for which the base field k is not necessarily algebraically closed (see Definition 2.1).

(a) *if* $G \subset \mathrm{PGL}_2(\mathbb{C})$ [2]*, then* G *has a regularly parametric cover,*
(b) *if* $G \not\subset \mathrm{PGL}_2(\mathbb{C})$*, then even the set of all Galois covers* $X \to \mathbb{P}^1_{\mathbb{C}}$ *of group* G *and with at most* r_0 *branch points is not regularly parametric, for any* $r_0 \geq 0$*.*

Hence, for $G \not\subset \mathrm{PGL}_2(\mathbb{C})$, letting the branch point number grow provides an endless source of "new" Galois covers of $\mathbb{P}^1_{\mathbb{C}}$ of group G, i.e., not mere rational pullbacks of covers with a bounded branch point number, and so truly new candidates to be defined over $\mathbb{Q}$.

Both statements of Theorem 1.1 are non-trivial. The one showing that finite subgroups of $\mathrm{PGL}_2(\mathbb{C})$ have a regularly parametric cover over $\mathbb{C}$ (with at most 3 branch points) was proved in [Dèb18, Corollary 2.5], as a consequence of the *twisting lemma* and Tsen's theorem. The statement about "general" finite groups, those not contained in $\mathrm{PGL}_2(\mathbb{C})$, is a new result of this paper. In particular, it solves [Dèb18, Problem 2.14].

Here is a more precise version. Given an integer $r \geq 0$ and an r-tuple $\mathbf{C}$ of non-trivial conjugacy classes of G, denote the stack of regular Galois covers $X \to \mathbb{P}^1$ of group G with r branch points by $\mathsf{H}_{G,r}$, and the stack of those with ramification type $(r, \mathbf{C})$ by $\mathsf{H}_{G,r}(\mathbf{C})$ (see §2.1); these are the *Hurwitz stacks*. From Theorem 1.1, if $G \not\subset \mathrm{PGL}_2(\mathbb{C})$, the set

$$\mathsf{H}_{G,\leq r_0}(\mathbb{C}) = \bigcup_{r \leq r_0} \mathsf{H}_{G,r}(\mathbb{C})$$

is never regularly parametric ($r_0 \geq 1$). More precisely, we have the following:

Theorem 1.2. *Let* k *be an algebraically closed field of characteristic* 0 *and let* G *be a finite group, not contained in* $\mathrm{PGL}_2(\mathbb{C})$*. Fix an integer* $r_0 \geq 0$*. For every suitably large integer* R*, depending on* r_0*, there is a non-empty Hurwitz stack* $\mathsf{H}_{G,R}(\mathbf{C})$ *such that* **not all** k*-covers in* $\mathsf{H}_{G,R}(\mathbf{C})$ *are rational pullbacks of* k*-covers in* $\mathsf{H}_{G,\leq r_0}$*.*

Finite subgroups of $\mathrm{PGL}_2(\mathbb{C})$, which are excluded in Theorem 1.2, are the Galois groups of genus 0 Galois covers. We show further that, if we also exclude the Galois groups of genus 1 Galois covers, then the conclusion of Theorem 1.2 holds for **all** Hurwitz stacks $\mathsf{H}_{G,R}(\mathbf{C})$ with $\mathbf{C}$ of suitably large length R (see Theorem 3.1). In fact, this stronger conclusion can even be obtained without the extra genus 1 assumption as long as the field k is also assumed to be uncountable (see Remark 3.7).

A related result (see Theorem 3.8) provides the following more explicit conclusion under the stronger assumption that the finite group G has at least 5 maximal non-conjugate cyclic subgroups (finite subgroups of $\mathrm{PGL}_2(\mathbb{C})$ have at most 3): *given any integer* $r_0 \geq 0$*, for every suitably large even integer* R*, there is a non-empty Hurwitz stack* $\mathsf{H}_{G,R}(\mathbf{C})$ *such that* ***no*** k*-cover in* $\mathsf{H}_{G,R}(\mathbf{C})$ *is a rational pullback of some* k*-cover in* $\mathsf{H}_{G,\leq r_0}$*.*

If, instead of the stacks $\mathsf{H}_{G,r}$, we consider the smaller stacks $\mathsf{H}_{G,r}(\mathbf{C})$, we obtain the following striking conclusion.

Theorem 1.3. *Let* k *be an algebraically closed field of characteristic* 0*, let* G *be a finite group not contained in* $\mathrm{PGL}_2(\mathbb{C})$*, and let* $(R, \mathbf{C})$ *be a ramification type for* G *with* $R \geq 4$*. Then there exists a ramification type* $(R+1, \mathbf{D})$ *for* G *such that* $\mathsf{H}_{G,R+1}(\mathbf{D}) \neq \emptyset$ *and no* k*-cover in* $\mathsf{H}_{G,R+1}(\mathbf{D})$ *is a pullback of a* k*-cover in* $\mathsf{H}_{G,R}(\mathbf{C})$*.*

Remark 1.4. (a) The assumption that k is algebraically closed can be weakened in some of the results above to only assume that k is ample, or even arbitrary (of characteristic 0) in some situations; see Theorem 3.10. Recall that a field k is *ample* if every geometrically irreducible smooth k-curve has either zero or infinitely many k-rational points. Ample

[2] Above and throughout the paper, the condition "$G \subset \mathrm{PGL}_2(\mathbb{C})$" (resp., "$G \not\subset \mathrm{PGL}_2(\mathbb{C})$") really means that G is *isomorphic* (resp., is *not isomorphic*) to a subgroup of $\mathrm{PGL}_2(\mathbb{C})$.

fields include separably closed fields, Henselian fields, fields $\mathbb{Q}^{\mathrm{tot}\mathbb{R}}$, $\mathbb{Q}^{\mathrm{tot}p}$ of totally real or p-adic algebraic numbers. See, e.g., [Jar11, BSF13, Pop14] for more on ample fields.

(b) Theorem 1.3 also holds for $R = 3$. The proof uses the same tools and techniques as for the case $R \geq 4$, but is longer and more technical; it is given in [DKLN18].

1.2. **Application.** Theorem 1.2 has the following consequence. Given an algebraically closed field k of characteristic 0, denote the set of all Galois covers $X \to \mathbb{P}^1_k$ of group G by $\mathsf{H}_G(k)$. Say that a finite group G has the *Beckmann–Black regular lifting property* over k if, for any two g_1 and g_2 in $\mathsf{H}_G(k)$, there exist $f \in \mathsf{H}_G(k)$ and two non-constant rational maps $T_{01}, T_{02} : \mathbb{P}^1_k \to \mathbb{P}^1_k$ such that $g_i = f_{T_{0i}}$ $(i = 1, 2)$.

Corollary 1.5. *Let k be an algebraically closed field of characteristic 0. The finite subgroups of $\mathrm{PGL}_2(\mathbb{C})$ are exactly those finite groups for which the Beckmann–Black regular lifting property over k holds.*

The proof combines Theorem 1.2 with [Dèb18, Theorem 2.1], and is given in §2.3.2 where the latter is recalled.

Our lifting property is a geometric variant of the Beckmann–Black *arithmetic* lifting property for which the cover $f \in \mathsf{H}_G(k)$ is requested to be defined over $\mathbb{Q}$ and to *specialize* to some given Galois extensions $E_1/\mathbb{Q}, \ldots, E_N/\mathbb{Q}$ of group G at some points $t_{01}, \ldots, t_{0N} \in \mathbb{P}^1(\mathbb{Q})$ [3]. There is no known counter-example to the latter. In comparison, our geometric variant is obvious for $N = 1$ (by picking $T_0(U) = U$ so that $f_{T_0} = f$), and Corollary 1.5 shows that it fails for $N \geq 2$ if $G \not\subset \mathrm{PGL}_2(\mathbb{C})$.

An intermediate stage towards counter-examples to the Beckmann–Black arithmetic property is to find groups G *with no $\mathbb{Q}$-parametric covers*, i.e. such that no cover $f \in \mathsf{H}_G(\mathbb{Q})$ specializes to all Galois extensions of $\mathbb{Q}$ of group G. First examples were given in [KL18, KLN19]. Extending them to all finite subgroups $G \not\subset \mathrm{PGL}_2(\mathbb{C})$ is a next challenge, to which we will devote a subsequent work. Theorem 1.3 will be a key ingredient, the central idea being to combine the strong non regular parametricity conclusions of Theorem 1.3 with a strategy from [Dèb18] designed to deduce non $\mathbb{Q}$-parametricity conclusions.

1.3. **Methods and organization of the paper.** Riemann's existence theorem (RET) is the fundamental theorem the above results build on. Furthermore, our proof of Theorem 1.2 exploits the geometric structure of the Hurwitz moduli spaces $\mathcal{H}_{G,R}(\mathbf{C})$, namely, it shows that the dimension of the subset of all covers obtained by pulling back a cover in $\mathsf{H}_{G,\leq r_0}$ is strictly smaller than that of $\mathcal{H}_{G,R}(\mathbf{C})$, for sufficiently large R.

The tools used to bound the dimension of the subset of pullbacks include bounding the degree of defining polynomials for covers using a Riemann–Roch based result of Sadi [Sad99], Chevalley's theorem, and combinatorial ramification arguments. For the analogous result over ample fields k, we show that $\mathsf{H}_{G,R}(\mathbf{C})(k)$ is non-empty, using Pop's $\frac{1}{2}$-Riemann existence theorem [Pop94], thus Zariski-dense in at least one connected component of $\mathcal{H}_{G,R}(\mathbf{C})$; and hence this set is not contained in the above smaller dimension subset of pullbacks. Over arbitrary fields of characteristic 0, we extend the theorem to certain families of groups using the rigidity method.

On the other hand, the proofs of Theorems 1.3 and 3.8 construct explicit ramification types whose Hurwitz stacks are non-empty by RET, and contain none of the pullbacks in question, as shown using combinatorial arguments, the Riemann–Hurwitz formula, and

[3] The case $N = 1$ is particularly significant as it supports Hilbert's strategy to solve the Inverse Galois Problem by first producing a $\mathbb{Q}$-regular Galois cover $f : X \to \mathbb{P}^1_{\mathbb{Q}}$ of given group. It is known to hold for some groups: abelian, S_n, A_n, dihedral of order $2n$ with $n > 1$ odd, etc.

Abhyankar's lemma. For an analogous result to Theorem 3.8 over ample fields k, we use $\frac{1}{2}$-RET once more to ensure that $\mathsf{H}_{G,R}(\mathbf{C})(k) \neq \emptyset$ for the constructed ramification types.

See §3 for the proof of Theorem 1.2 and its variants. See §4 for the proof of Theorem 1.3. §2 is a preliminary section providing the basic notation and terminology together with some general prerequisites.

2. Notation, terminology, and prerequisites

2.1. **Basic terminology** (for more details, see [DD97b] and [DL13]). The base field k is always assumed to be of characteristic 0. We also fix a big algebraically closed field containing k and the indeterminates that will be used, and in which every field compositum should be understood.

2.1.1. *Covers.* Given a field k, a ***k-variety*** is a geometrically irreducible and geometrically reduced quasiprojective k-scheme. A ***k-curve*** is a k-variety of dimension 1.

A field extension $F/k(T)$ is ***k-regular*** if $F \cap \overline{k} = k$. A ***k-regular cover*** $f : X \to \mathbb{P}^1_k$ is a non-constant finite morphism with X a smooth k-curve; the function field extension $k(X)/k(T)$ is then k-regular. If in addition $k(X)/k(T)$ is Galois, then $f : X \to \mathbb{P}^1_k$ is called a ***k-regular Galois cover***. If k is algebraically closed, we sometimes omit the word "k-regular".

We also use ***affine equations***: we mean the irreducible polynomial $P \in k[T, Y]$ of a primitive element of $k(X)/k(T)$, integral over $k[T]$. We say ***defining equation*** if the primitive element is not necessarily integral over $k[T]$; then $P \in k(T)[Y]$.

By ***group*** and ***branch point set*** of a k-regular cover $f : X \to \mathbb{P}^1_k$, we mean those of the extension $\overline{k}(X)/\overline{k}(T)$ [4]: the *group* of $\overline{k}(X)/\overline{k}(T)$ is the Galois group of its Galois closure and the *branch point set* of $\overline{k}(X)/\overline{k}(T)$ is the (finite) set of points $t \in \mathbb{P}^1(\overline{k})$ such that the associated discrete valuations are ramified in $\overline{k}(X)/\overline{k}(T)$.

The field k being of characteristic 0, we also have the ***inertia canonical invariant*** $\mathbf{C}$ of the k-regular cover $f : X \to \mathbb{P}^1_k$, defined as follows. If $\mathbf{t} = \{t_1, \ldots, t_r\}$ is the branch point set of f, then $\mathbf{C}$ is an r-tuple $(C_1, \ldots, C_r)$ of conjugacy classes of the group G of $\overline{k}(X)/\overline{k}(T)$: for $i = 1, \ldots, r$, the class C_i is the conjugacy class of the distinguished[5] generators of the inertia groups $I_{\mathfrak{P}}$ above t_i in the Galois closure of $\overline{k}(X)/\overline{k}(T)$. The pair $(r, \mathbf{C})$ is called the ***ramification type*** of f. More generally, given a finite group G, we say that a pair $(r, \mathbf{C})$ is ***a ramification type*** for G over k if it is the ramification type of at least one k-regular Galois cover $f : X \to \mathbb{P}^1_k$ of group G.

We also use the notation $\mathbf{e} = (e_1, \ldots, e_r)$ for the r-tuple with ith entry the ramification index $e_i = |I_{\mathfrak{P}}|$ of primes above t_i; e_i is also the order of elements of C_i, $i = 1, \ldots, r$.

We say that two k-regular covers $f : X \to \mathbb{P}^1_k$ and $g : Y \to \mathbb{P}^1_k$ are ***$\mathbb{P}^1_k$-isomorphic*** if there is an isomorphism $\chi : X \to Y$ defined over k such that $f = g \circ \chi$.

2.1.2. *Hurwitz stacks and Hurwitz spaces.* Given a finite group G, an integer $r \geq 1$, an r-tuple $\mathbf{C}$ of non-trivial conjugacy classes of G, and a field k (of characteristic zero), we use the following notation:

- $\mathsf{H}_G(k)$: set of all k-regular Galois covers $f : X \to \mathbb{P}^1_k$ of group G,
- $\mathsf{H}_{G,r}(k)$ (resp., $\mathsf{H}_{G,\leq r}(k)$): subset of $\mathsf{H}_G(k)$ defined by the extra condition that the branch point number is r (resp., that the branch point number is $\leq r$),
- $\mathsf{H}_{G,r}(\mathbf{C})(k)$: subset of $\mathsf{H}_{G,r}(k)$ defined by the extra condition that the inertia canonical invariant is $\mathbf{C}$.

[4] which is the function field extension associated with $f \otimes_k \overline{k} : X \otimes_k \overline{k} \to \mathbb{P}^1_{\overline{k}}$.

[5] in the sense that they correspond to $e^{2i\pi/e_i}$ in the canonical isomorphism $I_{\mathfrak{P}} \to \mu_{e_i} = \langle e^{2i\pi/e_i} \rangle$.

The sets $\mathsf{H}_{G,r}(k)$ and $\mathsf{H}_{G,r}(\mathbf{C})(k)$ can be viewed as the sets of k-rational points on some *stacks* $\mathsf{H}_{G,r}$ and $\mathsf{H}_{G,r}(\mathbf{C})$, usually called ***Hurwitz stacks***. More formally, $\mathsf{H}_{G,r}(k)$ is the category whose objects are the k-regular Galois covers $f : X \to \mathbb{P}^1_k$ with r branch points and given with an isomorphism $G \to \mathrm{Gal}(k(X)/k(T))$, and morphisms are the $\mathbb{P}^1_k$-isomorphisms commuting with the action of G; and similarly for $\mathsf{H}_{G,r}(\mathbf{C})$.

We use the phrase *sets of k-points on the Hurwitz stacks* $\mathsf{H}_{G,r}$ *and* $\mathsf{H}_{G,r}(\mathbf{C})$ for the sets $\mathsf{H}_{G,r}(k)$ and $\mathsf{H}_{G,r}(\mathbf{C})(k)$, but shall not use the stack structure.

On the other hand, we shall need the structure of variety of the associated moduli spaces, notably in §3.1 for the proof of Theorem 1.2. The stacks $\mathsf{H}_{G,r}$ and $\mathsf{H}_{G,r}(\mathbf{C})$ have indeed a coarse moduli space, which we denote by $\mathcal{H}_{G,r}$ and $\mathcal{H}_{G,r}(\mathbf{C})$, respectively. They are commonly referred to as ***Hurwitz spaces***; see [FV91] for more details[6]. They are finite unions of r-dimensional varieties and have this property: if k is an algebraically closed field of characteristic 0, the sets $\mathcal{H}_{G,r}(k)$ and $\mathcal{H}_{G,r}(\mathbf{C})(k)$ are in one-one correspondence with the sets of isomorphism classes of objects in the categories $\mathsf{H}_{G,r}(k)$ and $\mathsf{H}_{G,r}(\mathbf{C})(k)$, respectively. Finally, if k is a not necessarily algebraically closed field (but is still of characteristic 0) and $f \in \mathsf{H}_{G,r}(k)$, then its isomorphism class $[f]$ still corresponds to a k-rational point of $\mathcal{H}_{G,r}(\mathbf{C})$. If additionally G has trivial center, then the spaces $\mathcal{H}_{G,r}(\mathbf{C})$ are in fact fine moduli spaces, whence conversely any k-rational point on $\mathcal{H}_{G,r}(\mathbf{C})$ corresponds to a k-regular Galois cover.

2.2. **Pullback and regular parametricity.** Let k be a field of characteristic zero and $f : X \to \mathbb{P}^1_k$ a k-regular cover. Let $T_0 \in k(U) \setminus k$; we make no distinction between the rational function T_0 and the rational map $T_0 : \mathbb{P}^1_k \to \mathbb{P}^1_k$ [7]. The fiber product $X \times_{f,T_0} \mathbb{P}^1_k$ provides a cartesian square

$$\begin{array}{ccc} X \times_{f,T_0} \mathbb{P}^1_k & \longrightarrow & \mathbb{P}^1_k \\ \downarrow & & \downarrow {\scriptstyle T_0} \\ X & \xrightarrow{\ f\ } & \mathbb{P}^1_k \end{array}$$

When $X \times_{f,T_0} \mathbb{P}^1_k$ is geometrically irreducible, denote by $f_{T_0} : X_{T_0} \to \mathbb{P}^1_k$ the smooth projective model of the top horizontal map: X_{T_0} is the normalization of $\mathbb{P}^1_k$ in the function field of $X \times_{f,T_0} \mathbb{P}^1_k$. The map $f_{T_0} : X_{T_0} \to \mathbb{P}^1_k$ is then a k-regular cover, which we call the ***pullback of** f **along*** T_0. More generally, covers f_{T_0} are called ***rational pullbacks of*** f. If $f : X \to \mathbb{P}^1_k$ is additionally assumed to be Galois of group G, then $f_{T_0} : X_{T_0} \to \mathbb{P}^1_k$ remains a k-regular Galois cover of group G.

Given an affine equation $P \in k[T,Y]$ of f, consider the subset $H_{P,\overline{k}(U)}$ of $\overline{k}(U)$ of all T_0 such that $P(T_0(U),Y)$ is irreducible in $\overline{k}(U)[Y]$. It is a Hilbert subset of $\overline{k}(U)$ [FJ08, Chapter 12]. If $T_0 \in H_{P,\overline{k}(U)} \cap k(U)$, then $T_0 \notin k$ and the fiber product $X \times_{f,T_0} \mathbb{P}^1_k$ is geometrically irreducible; hence the pullback f_{T_0} is a k-regular cover and $P(T_0(U),Y)$ is a defining equation of f_{T_0}. The subset $H_{P,\overline{k}(U)} \cap k(U)$ only depends on f (and not on the specific equation $P(T,Y)$). Denote it by $H_{f,k}$. The field $\overline{k}(U)$ being Hilbertian [FJ08, Proposition 13.2.1], the Hilbert subset $H_{P,\overline{k}(U)}$ is "big" in various senses: by definition of a Hilbertian field, it is infinite; by Theorems 3.3 and 3.4 from [Dèb99], it is dense for the Strong Approximation Topology. The same is true of the set $H_{f,k}$.

[6] Due to our definition of the categories $\mathsf{H}_{G,r}(k)$ and $\mathsf{H}_{G,r}(\mathbf{C})(k)$, it is the so-called *inner* version of Hurwitz spaces that we shall be working with.

[7] In particular, the degree of $T_0 \in k(U)$ (the maximum of numerator degree and denominator degree in coprime notation) is the same as the degree of the associated map $\mathbb{P}^1_k \to \mathbb{P}^1_k$.

Definition 2.1. For $\mathsf{H} \subset \mathsf{H}_G(k)$, we define

$$\mathrm{PB}(\mathsf{H}) = \left\{ f_{T_0} \,\middle|\, \begin{array}{l} f \in \mathsf{H} \\ T_0 \in H_{f,k} \end{array} \right\} \quad \text{(PB for "PullBack").}$$

A subset H of $\mathsf{H}_G(k)$ is *k-regularly parametric* if $\mathrm{PB}(\mathsf{H}) \supset \mathsf{H}_G(k)$. We say that a cover $f \in \mathsf{H}_G(k)$ is *k-regularly parametric* if the subset $\{f\}$ of $\mathsf{H}_G(k)$ is.

The k-regularly parametricity notion relates to the classical notion of *genericity* (in one parameter; see, e.g., [JLY02]): clearly, if a cover $f \in \mathsf{H}_G(k)$ is generic, then it is k-regularly parametric. The paper [DKLN20] says more about how the two notions compare.

2.3. **Prerequisites.** Let k be an algebraically closed field of characteristic 0.

2.3.1. *Riemann Existence Theorem.* This fundamental tool of the theory of covers of $\mathbb{P}^1$ allows turning questions about covers into combinatorics and group theory considerations.

Riemann Existence Theorem (RET). *Given a finite group G, an integer $r \geq 2$, a subset $\mathbf{t}$ of $\mathbb{P}^1(k)$ of r points, and an r-tuple $\mathbf{C} = (C_1, \dots, C_r)$ of non-trivial conjugacy classes of G, there is a Galois cover $f : X \to \mathbb{P}^1_k$ of group G, branch point set $\mathbf{t}$, and inertia canonical invariant $\mathbf{C}$ if and only if there exists $(g_1, \dots, g_r) \in C_1 \times \cdots \times C_r$ such that $g_1 \cdots g_r = 1$ and $\langle g_1, \dots, g_r \rangle = G$. Furthermore, the number of such covers $f : X \to \mathbb{P}^1_k$, counted up to $\mathbb{P}^1_k$-isomorphism classes, equals the number of r-tuples $(g_1, \dots, g_r)$ as above, counted modulo componentwise conjugation by an element of G.*

Cf., e.g., [Völ96, Theorem 2.13] for the mere existence statement, or [Dèb01] for a detailed overview.

The RET shows that a pair $(r, \mathbf{C})$ is a ramification type for G over k if the set, traditionally called the ***Nielsen class***, of all $(g_1, \dots, g_r) \in C_1 \times \cdots \times C_r$ such that $g_1 \cdots g_r = 1$ and $\langle g_1, \dots, g_r \rangle = G$ is non-empty. We shall use the RET to construct Galois covers of given group G and with some special ramification type.

2.3.2. *Bounds for the branch point number and the genus of pulled-back covers.*

Theorem 2.2. *Let $f : X \to \mathbb{P}^1_k \in \mathsf{H}_G(k)$ and T_0 be in the Hilbert subset $H_{f,k}$. Denote the branch point number of f (resp., f_{T_0}) by r (resp., r_{T_0}) and the genus of X (resp., X_{T_0}) by g (resp., g_{T_0}). Then $r \leq r_{T_0}$ and $g \leq g_{T_0}$. Moreover, if $g > 1$ and T_0 is not an isomorphism, then $g < g_{T_0}$.*

Proof. For branch point numbers, see [Dèb18, Theorem 2.1]. Regarding genera, we may assume $g \neq 0$ and T_0 is not an isomorphism. The claim then follows from applying the Riemann–Hurwitz formula to the cover $X_{T_0} \to X$. Namely, we obtain $2g_{T_0} - 2 \geq N(2g-2)$ with $N = \deg(f)$, whence $g_{T_0} \geq 2(g-1)+1 \geq g$ if $g \geq 1$, with equality only if $g = 1$. □

We can now explain how Corollary 1.5 is deduced from Theorem 1.2.

Proof of Corollary 1.5 assuming Theorem 1.2. First, assume that the group G satisfies the Beckmann–Black regular lifting property over k. Pick a Galois cover $g_1 \in \mathsf{H}_G(k)$ (such a cover exists from the RET). Let r_1 be the branch point number of g_1. Then, for any $g_2 \in \mathsf{H}_G(k)$, there exists $f \in \mathsf{H}_G(k)$ and $T_{01}, T_{02} \in k(U)$ such that $g_i = f_{T_{0i}}$ $(i = 1, 2)$. From Theorem 2.2, it follows from $g_1 = f_{T_{01}}$ that the branch point number of f is $\leq r_1$. This shows that $\mathsf{H}_{G, \leq r_1}(k)$ is regularly parametric. From Theorem 1.2, $G \subset \mathrm{PGL}_2(\mathbb{C})$.

Conversely, if $G \subset \mathrm{PGL}_2(\mathbb{C})$, then G has a k-regularly parametric cover $f : X \to \mathbb{P}^1_k$ (see [Dèb18, Corollary 2.5]); *a fortiori* the Beckmann–Black regular lifting property holds over k. □

Remark 2.3. The proofs above of Theorem 2.2 and Corollary 1.5 cite two results which are only stated with $k = \mathbb{C}$: Theorem 2.1 and Corollary 2.5 from [Dèb18]. These two results however hold more generally over any algebraically closed field of characteristic 0. Indeed, regarding [Dèb18, Theorem 2.1], the assumption $k = \mathbb{C}$ was made for simplicity and can readily be generalized. As to [Dèb18, Corollary 2.5], the main ingredient there is Tsen's theorem that the field $\mathbb{C}(U)$ is quasi-algebraically closed, which is also true with $\mathbb{C}$ replaced by any algebraically closed field.

2.3.3. *On varieties and their dimension.* In the following, we recall some well-known facts from algebraic geometry about the structure and dimension of images and preimages under algebraic morphisms.

It is elementary that the image of an n-dimensional variety under an algebraic morphism is always of dimension $\leq n$. A bound in the opposite direction is given via the dimension of a fiber (see, e.g., [Mum99, §1.8, Theorems 2 and 3]):

Theorem 2.4. *Let $f : X \to Y$ be a dominant morphism between varieties X and Y. For any point $p \in f(X)$, we have* $\dim(Y) \leq \dim(X) \leq \dim(Y) + \dim(f^{-1}(p))$.

We refer to, e.g., [Gro64, p. 239, lemme 1.8.4.1] for the next theorem:

Theorem 2.5 (Chevalley)**.** *Let $f : X \to Y$ be a morphism between varieties X and Y. Then the image of any constructible subset of X is constructible*[8].

In particular, the image of any subvariety X_0 of X is a finite union of varieties $Y_i \subset Y$, each of dimension at most $\dim(X_0)$.

For short, we shall say that a subset S of a variety X is *of dimension* $\leq d$, if it is contained in a finite union of subvarieties of dimension $\leq d$.

2.3.4. *Defining equations for Galois covers and their pullbacks.* To prove Theorem 1.2, we shall use affine and defining equations of Galois covers of $\mathbb{P}^1_k$ (as defined in §2.1.1).

Lemma 2.6. *Let G be a finite group and $r_0 \in \mathbb{N}$. Then every Galois cover in $\mathsf{H}_{G,\leq r_0}(k)$ can be defined by an affine equation $P(T, Y) = 0$, where $P \in k[T][Y]$ is irreducible, monic in Y, of bounded T-degree depending only on r_0 and G, and of degree $|G|$ in Y.*

Proof. The fact that every Galois cover of group G and of bounded genus $g \leq g_0$ can be defined by an affine equation of bounded T-degree depending only on g_0 and $|G|$ follows essentially from an application of the Riemann–Roch theorem. Concretely, in [Sad99, §2.2] the upper bound $(2g_0 + 1)|G| \log |G|/ \log(2)$ was obtained, cf. also [Dèb17, Lemma 4.1]. It then suffices to note that the genus g of a Galois cover of $\mathbb{P}^1_k$ with r_0 branch points is bounded from above only in terms of r_0 and G: the Riemann–Hurwitz formula gives $2g \leq 2 - 2|G| + r_0(|G| \cdot (1 - 1/e_{\max}))$, with $e_{\max}$ the maximal element order in G. □

Definition 2.7. Let $d, e \in \mathbb{N}$. We denote by $\mathcal{P}_{d,e}$ the space of polynomials $P(T, Y) \in k[T, Y]$ of degree exactly d in T and exactly e in Y, viewed up to multiplicative constants. Similarly, let $\mathcal{P}_{\leq d}$ denote the space of polynomials $Q(T) \in k[T]$ of degree at most d, viewed up to multiplicative constants. Furthermore, denote by $\mathcal{R}_d$ the set of rational functions over k in one indeterminate U of degree exactly d.

The spaces $\mathcal{P}_{d,e}$ and $\mathcal{P}_{\leq d}$ are varieties in a natural way, via identifying the polynomials $P(T, Y) = \sum_{i=0}^{d} \sum_{j=0}^{e} \alpha_{i,j} T^i Y^j$ and $Q(T) = \sum_{i=1}^{d} \beta_i T^i$ with the coordinate tuples $(\alpha_{i,j})_{i,j}$ and $(\beta_i)_i$, respectively, in the corresponding projective space. Similarly, $\mathcal{R}_d$ is a variety by

[8]Here, a subset of a topological space is called *constructible* if it is a finite union of locally closed sets.

identifying a rational function $T_0 = T_0(U) = (\sum_{i=0}^{d} \beta_i U^i)/(\sum_{j=0}^{d} \gamma_j U^j)$, with coprime numerator and denominator, with the coordinate tuple $(\beta_0 : \cdots : \beta_d : \gamma_0 : \cdots : \gamma_d) \in \mathbb{P}^{2d+1}$. Note that the degree and coprimeness assumptions fix the numerator and denominator up to constant factor, whence the above identification is well-defined.

We can now define pullback maps on the level of the above spaces $\mathcal{P}_{d,e}$ and $\mathcal{R}_d$:

Lemma 2.8. *Let d_1, d_2, and d_3 be positive integers. Then the map*

$$\widetilde{\mathrm{PB}} : \mathcal{P}_{d_1,d_2} \times \mathcal{R}_{d_3} \to \mathcal{P}_{d_1 \cdot d_3, d_2},$$

defined by $(P(T,Y), T_0(U)) \mapsto$ "numerator of $P(T_0(U), Y)$", is a morphism of algebraic varieties.

Proof. Let $P(T,Y) = \sum_{i=0}^{d_1} \sum_{j=0}^{d_2} \alpha_{i,j} T^i Y^j$ and $T_0 = (\sum_{k=0}^{d_3} \beta_k U^k)/(\sum_{k=0}^{d_3} \gamma_k U^k)$. Then

$$\widetilde{\mathrm{PB}}(P, T_0) = \sum_{i,j} \alpha_{i,j} (\sum_k \beta_k U^k)^i (\sum_k \gamma_k U^k)^{d_1 - i} Y^j,$$

and identification with the spaces of coordinate tuples shows that $\widetilde{\mathrm{PB}}$ is given by a polynomial map. □

3. Proofs of Theorem 1.2 and its variants

In §1.1, we mention two variants of Theorem 1.2: a "genus ≥ 2" version (Theorem 3.1) and an "explicit" variant (Theorem 3.8). §3.1 is devoted to the proof of the former, the proof of how Theorem 1.2 can be deduced being explained at the end of §3.1.1 (where Theorem 3.1 is stated). §3.2 is devoted to the proof of Theorem 3.8. Throughout this section, except in §3.3, the field k is algebraically closed of characteristic 0. Our proofs make use of the fact that k is algebraically closed but certain parts carry over to more general fields. We collect such considerations in §3.3.

3.1. **Proof of Theorem 1.2.**

3.1.1. *Reduction to Theorem 3.1 and Lemma 3.2.* The following theorem is our strongest result regarding pullbacks of Galois covers of genus at least 2. Its proof is also the main part of the proof of Theorem 1.2.

Theorem 3.1. *Let G be a finite group, let $r_0 \in \mathbb{N}$, and let*

$$\mathsf{H}_{G,\leq r_0, g\geq 2}(k) = \mathsf{H}_{G,\leq r_0}(k) \setminus \{genus \leq 1 \ covers\}$$

be the set of all Galois covers $f : X \to \mathbb{P}^1_k$ of genus at least 2 with Galois group G and at most r_0 branch points. Then there exists $R_0 \in \mathbb{N}$ such that, for every ramification type $(R, \mathbf{C})$ for G with $R \geq R_0$, we have

$$\mathsf{H}_{G,R}(\mathbf{C})(k) \not\subset \mathrm{PB}(\mathsf{H}_{G,\leq r_0, g\geq 2}(k)).$$

In particular, $\mathsf{H}_{G,\leq r_0,g\geq 2}(k)$ is not k-regularly parametric.

Theorem 3.1 is proved in §3.1.2-4 below. The following lemma shows non-parametricity for sets of Galois k-covers of genus 1:

Lemma 3.2. *Let $\mathsf{H}_{G,g=1}(k)$ be the set of Galois covers $f : X \to \mathbb{P}^1_k$ of genus 1 with group G. Then there exists a ramification type $(r, \mathbf{C})$ for G such that no cover in $\mathsf{H}_{G,r}(\mathbf{C})(k)$ is a pullback from any cover in $\mathsf{H}_{G,g=1}(k)$. In particular, $\mathsf{H}_{G,g=1}(k)$ is not k-regularly parametric.*

Proof. Let $f : X \to \mathbb{P}^1 \in \mathsf{H}_{G,g=1}(k)$. As a consequence of the Riemann–Hurwitz formula, the tuple of element orders in the inertia canonical invariant of f is one of $(2,2,2,2)$, $(3,3,3)$, $(2,4,4)$, or $(2,3,6)$. Furthermore, in each case, G has a normal subgroup N with cyclic quotient group G/N of order 2, 3, 4 and 6, respectively, and such that the quotient map $X \to X/N$ is an unramified cover of genus-1 curves over k. Assume first $|N| = 1$. Then G is cyclic[9] and, therefore, there exist Galois covers of $\mathbb{P}^1_k$ of group G and genus 0. In particular, no set of covers of genus ≥ 1 can have those as pullbacks, by Theorem 2.2.

Assume therefore $|N| > 1$. Let $x \in N \setminus \{1\}$, and let $(r, \mathbf{C})$ be any ramification type for G involving the conjugacy class of x. Since $X \to X/N$ is unramified, its image under any rational pullback of f must also be unramified. But, of course, for any cover $\widetilde{X} \to \mathbb{P}^1$ with inertia canonical invariant $\mathbf{C}$, the subcover $\widetilde{X} \to \widetilde{X}/N$ is ramified by definition. Therefore, no cover of inertia canonical invariant $\mathbf{C}$ can be a pullback of f. □

Remark 3.3. In the case that G is non-cyclic, the above proof shows immediately that, for $(r, \mathbf{C})$ any ramification type of genus 1 with group G ($r \in \{3,4\}$) and for each $s \geq r+1$, there exists a ramification type $(s, \mathbf{D})$ for G such that no k-cover in $\mathsf{H}_{G,s}(\mathbf{D})$ is a pullback of some k-cover in $\mathsf{H}_{G,r}(\mathbf{C})$. Indeed, for the only critical case $s = r+1$, it suffices to replace $(x_1, \dots, x_n) \in \mathbf{C}$, where $x_1 \notin N$ without loss, by $(x_0, x_0^{-1}x_1, \dots, x_n)$ with $x_0 \in N \setminus \{1\}$.

Assuming Theorem 3.1 and Lemma 3.2, we can now derive Theorem 1.2.

Proof of Theorem 1.2. By assumption, there is no Galois cover of $\mathbb{P}^1$ of group G and genus 0. Let $(R, \mathbf{C})$ be a ramification type for G with $\mathbf{C} = (C_1, \dots, C_R)$. From Theorem 3.1, we know that not all covers in $\mathsf{H}_{G,R}(\mathbf{C})(k)$ are rational pullbacks of some element of $\mathsf{H}_{G,\leq r_0}(k)$ of genus ≥ 2, if the length R of $\mathbf{C}$ is sufficiently large (depending on r_0). From Lemma 3.2 and its proof, we know that no k-cover in $\mathsf{H}_{G,R}(\mathbf{C})$ is a rational pullback of some k-cover in $\mathsf{H}_{G,\leq r_0}$ of genus 1, if $\mathbf{C}$ contains certain conjugacy classes. Altogether, if $\mathbf{C}$ contains all classes of G sufficiently often, then certainly not all k-covers in $\mathsf{H}_{G,R}(\mathbf{C})$ are reached via rational pullback of some k-cover in $\mathsf{H}_{G,\leq r_0}$. □

3.1.2. *Proof of Theorem 3.1: some dimension estimates.* To prepare the proof of Theorem 3.1, we investigate the behaviour of rational pullbacks of Galois covers.

Recall that we have introduced two different ways of associating algebraic varieties to certain sets of Galois covers: the Hurwitz spaces and the spaces of defining equations. In the next lemma, we relate both concepts via a dimension estimate stating, in particular, that, in order to obtain defining equations for all covers in an r-dimensional Hurwitz space, we require at least r-dimensional subvarieties in the space of defining equations.

To state the lemma, denote by $\mathcal{P}_{d,e}^{\text{sep}}$ the subset of separable (in Y) polynomials in $\mathcal{P}_{d,e}$ (the latter set is introduced in Definition 2.7). Note that this is a dense open subset of $\mathcal{P}_{d,e}$. Due to Lemma 2.6, when looking for defining equations for covers in some $\mathsf{H}_{G,\leq r_0}(k)$, we can restrict without loss to a suitable finite union of $\mathcal{P}_{d,e}^{\text{sep}}$ (for d smaller than some bound depending only on G and r_0, and in fact always with $e = |G|$).

Lemma 3.4. *Let $r, s, d, e \in \mathbb{N}$. Let $V \subset \mathcal{P}_{d,e}^{\text{sep}}$ be a subvariety of dimension s. Let G be a finite group and let $(r, \mathbf{C})$ be a ramification type for G. Then the set of $f \in \mathcal{H}_{G,r}(\mathbf{C})(k)$ which have a defining equation in V is of dimension $\leq s$* [10]*. In particular, if $s < r$, there are infinitely many covers in $\mathsf{H}_{G,r}(\mathbf{C})(k)$ which do not have a defining equation in V.*

[9] In fact, $|G| \in \{2,3,4,6\}$, since G is then a group of automorphisms of some elliptic curve, see [Sil09, Chapter III, Theorem 10.1].

[10] Note here that equivalent covers have the same defining equations by definition, so that the term "defining equation for an element of $\mathcal{H}_{G,r}(\mathbf{C})(k)$" is indeed well-defined.

Proof. Denote the discriminant of a polynomial $Q(Y) = \sum_{i=0}^{e} a_i Y^i$ by

$$\Delta(Q) = a_e^{2e-2} \prod_{i<j} (r_i - r_j)^2,$$

where the r_i's are the roots of Q, counted with multiplicities. The discriminant induces an algebraic morphism $\Delta : \mathcal{P}_{d,e}^{\text{sep}} \to \mathcal{P}_{\le c}$, $P(T,Y) \mapsto \Delta(P)$ (where P is viewed as a polynomial in Y) to the space of polynomials in T of degree $\le c$ up to constant factors (see Definition 2.7). Note that indeed the image of a separable polynomial is nonzero. The fact that the degree of $\Delta(P)$ is bounded only in terms of d and e follows easily from the fact that the discriminant is a polynomial expression in the coefficients, viewed as transcendentals. In particular, the dimension of $\Delta(V) \subseteq \mathcal{P}_{\le c}$ is at most $\dim(V) = s$.

Next, for any $r \le t \le c$ and any r-subset R of $\{1,\dots,t\}$, consider the morphisms $u : (\mathbb{A}^1)^t \to \mathcal{P}_{\le c}$ given by $(a_1,\dots,a_t) \mapsto \prod_{i=1}^t (T - a_i)$ and $v : (\mathbb{A}^1)^t \to (\mathbb{A}^1)^r$ the projection on the coordinates in R. For each of these finitely many possible maps u, v, the map u is finite and so $v(u^{-1}(W))$ is of dimension $\le s$, where $W = \Delta(V)$. But, since any branch point (assumed to be finite without loss) of a Galois cover of $\mathbb{P}_k^1$ is necessarily a root of the discriminant of a defining equation, such a cover can only have a defining equation in V if its branch point set is in $v(u^{-1}(W))$ for some u, v as above. Now, let $\mathcal{U}^r$ and $\mathcal{U}_r$ denote the spaces of ordered and unordered r-sets in $\mathbb{P}^1$, respectively. There is a well-defined finite morphism from $\mathcal{H}_{G,r}(\mathbf{C})$ to $\mathcal{U}_r$: the branch point reference map. Now, let $\mathcal{H}' = \mathcal{H}_{G,r}(\mathbf{C}) \times_{\mathcal{U}_r} \mathcal{U}^r$ be the "ordered branch point set version" of the Hurwitz space $\mathcal{H}_{G,r}(\mathbf{C})$. Then the branch point reference map induces a finite morphism $\psi : \mathcal{H}' \to \mathcal{U}^r (\subset (\mathbb{P}^1)^r)$. Furthermore, there is a natural finite morphism $\pi : \mathcal{H}' \to \mathcal{H}_{G,r}(\mathbf{C})$.

In particular, each set $\pi(\psi^{-1}(v(u^{-1}(W))))$, and thus finally also the set of $f \in \mathcal{H}_{G,r}(\mathbf{C})(k)$ having (only finite branch points and) a defining equation in V, is of dimension $\le s$.

The additional assertion in the case $s < r$ follows immediately, since $\psi(\mathcal{H}') \cap (\mathbb{A}^1)^r$ is of dimension r and, in fact, equal to the set of all ordered r-sets in $\mathbb{A}^1$, by the Riemann Existence Theorem. □

In the proof of Theorem 3.1, we shall show as an intermediate result that a rational function pulling a prescribed Galois cover f of $\mathbb{P}_k^1$ back into a prescribed $\mathsf{H}_{G,R}(\mathbf{C})(k)$ can only have a certain maximal number of branch points outside of the branch point set of f. We then require the following auxiliary result stating that varieties of rational functions with such partially prescribed branch point sets cannot be too large.

Lemma 3.5. *Let $d, m, n, s \in \mathbb{N}$, let W be a subvariety of $\mathcal{P}_{m,n}^{sep}$, and let $\mathcal{R}_d$ be as in Definition 2.7. Then the set $W' \subset W \times \mathcal{R}_d$ of all $(P, T_0) \in W \times \mathcal{R}_d$ such that at most s branch points of T_0 are not roots of $\Delta(P)$ is of dimension at most* $\dim W + s + 3$.

Proof. Denote the discriminant map on $\mathcal{P}_{m,n}^{\text{sep}}$ by Δ_1 and the one on $\mathcal{R}_d$ by Δ_2. Here we define the discriminant of a rational function $T_0(U) = T_{0,1}(U)/T_{0,2}(U)$ as the discriminant of the polynomial $T_{0,1}(U) - T \cdot T_{0,2}(U)$ with respect to U. Note that, in the special case of rational functions, every root of the discriminant is in fact a branch point (see, e.g., [Mül02, Lemma 3.1] for a stronger version of this statement). This means that, with $u : (a_1,\dots,a_t) \mapsto \prod_{i=1}^t (T - a_i)$ as before, an element of $u^{-1}(\Delta_2(T_0))$ is already the exact branch point set of T_0, up to multiplicities.

Consider now the following chain of maps:

$$W' \subset W \times \mathcal{R}_d \xrightarrow{\text{id}\times\Delta_2} W \times \mathcal{P}_{\le t} \xleftarrow{\text{id}\times u} W \times (\mathbb{A}^1)^t \xleftarrow{\alpha} W \times (\mathbb{A}^1)^t,$$

where α is defined by $\alpha(P, (a_1,\dots,a_t)) = (P, (\Delta_1(P)(a_1),\dots,\Delta_1(P)(a_t)))$. Clearly, all maps in this chain are morphisms and, except for the first map $\text{id} \times \Delta_2$, they are all finite.

Therefore, $(\mathrm{id} \times \Delta_2)(W')$ is of the same dimension as $\alpha^{-1}\big((\mathrm{id} \times (u^{-1} \circ \Delta_2))(W')\big)$ and, by definition of W', the latter is contained in one of finitely many varieties isomorphic to $W \times (\mathbb{A}^1)^s$. Indeed, up to repetitions, all except for at most s roots of $\Delta_2(T_0)$ are mapped to 0 under $\Delta_1(P)$, for $(P, T_0) \in W'$. Thus, $(\mathrm{id} \times \Delta_2)(W')$ is of dimension at most $\dim(W) + s$. Theorem 2.4 then yields that W' is of dimension at most $\dim(W) + s + \dim(\Delta_2^{-1}(p))$ for any p equal to the discriminant of a rational function T_0 as above.

It remains to show that $\Delta_2^{-1}(p)$ is of dimension ≤ 3. Now, the set of genus zero covers $\mathbb{P}^1_k \to \mathbb{P}^1_k$ (viewed up to equivalence) of degree d with prescribed branch point set is finite, and each such cover is given by a degree-d rational function, unique up to $\mathrm{PGL}_2(k)$-equivalence. Since $\dim(\mathrm{PGL}_2) = 3$, the claim follows, completing the proof. □

3.1.3. *Proof of Theorem 3.1: reduction to Lemma 3.6.*

Lemma 3.6. *Let G be a finite group and $f : X \to \mathbb{P}^1_k$ a Galois cover with group G and genus ≥ 1. Then, for every $j \in \mathbb{N}$, there exists a constant $R_0 \in \mathbb{N}$, depending only on j and the branch point number of f, such that, for every class-R-tuple $\mathbf{C}$ of G ($R \geq R_0$) and for every rational function $T_0 : \mathbb{P}^1_k \to \mathbb{P}^1_k$ in $H_{f,k}$ with more than $R - j$ branch points outside the branch point set of f, the pullback of f along T_0 is not in $\mathsf{H}_{G,R}(\mathbf{C})(k)$.*

Proof of Theorem 3.1 assuming Lemma 3.6. Let $f : X \to \mathbb{P}^1 \in \mathsf{H}_{G,\leq r_0, g\geq 2}(k)$. Let g be the genus of X. Let $F = F(T,Y)$ be a separable defining equation for f, of minimal degree in T. Using the pullback map $\widetilde{\mathrm{PB}}$ as in Lemma 2.8, denote by $\widetilde{\mathrm{PB}}(f)$ the set of all pullbacks of F by rational functions of arbitrary degree, i.e., $\widetilde{\mathrm{PB}}(f) = \cup_{d\in\mathbb{N}} \widetilde{\mathrm{PB}}(\{F\} \times \mathcal{R}_d)$. Then $\widetilde{\mathrm{PB}}(f)$ contains defining equations for all rational pullbacks of the cover f.

Let $(R, \mathbf{C})$ be a ramification type for G. By the Riemann–Hurwitz formula, the genus of a Galois k-cover of group G arising as a degree-d pullback of f is $\geq d(g-1)+1$. As $g \geq 2$, this shows that there is $d_0 \in \mathbb{N}$, depending only on $\mathbf{C}$, such that, for all $d > d_0$, a degree-d pullback of f cannot have inertia canonical invariant $\mathbf{C}$ (as the genus is the same for all covers with invariant $\mathbf{C}$). In other words, to investigate the set of polynomials in $\widetilde{\mathrm{PB}}(f)$ which are defining equations for covers in $\mathsf{H}_{G,R}(\mathbf{C})(k)$, it suffices to restrict to pullback functions $T_0 \in \mathcal{R}_d$, $d \leq d_0$, with some bound $d_0 \in \mathbb{N}$ depending only on $\mathbf{C}$.

Let $D \in \mathbb{N}$ be such that every $f \in \mathsf{H}_{G,\leq r_0}(k)$ has a separable defining equation in some space $\mathcal{P}_{d_1,|G|}$ with $d_1 \leq D$. Such D exists by Lemma 2.6. Let δ be the dimension of $\mathcal{P}_{D,|G|}$ (to be explicit, $\delta = (D+1)(|G|+1) - 1$).

Fix $j > \delta + 3$, choose R_0 large enough (see the proof of Lemma 3.6 for an explicit bound) and $R \geq R_0$, and denote by $\mathcal{S}_f$ the set of $T_0 \in H_{f,k}$ for some $f \in \mathsf{H}_{G,\leq r_0, g\geq 2}(k)$ such that f_{T_0} has R branch points. As seen above, the degree of such T_0 is absolutely bounded from above (in terms of the genus, and thus the branch point number of f) and, by Lemma 3.6, all $T_0 \in \mathcal{S}_f$ have at most $R - j$ branch points outside the branch point set of f. *A fortiori*, they are in the set $\mathcal{S}'_F$ of rational functions (of bounded degree as before and) with at most $R - j$ finite branch points outside the set of roots of the discriminant of $F(T,Y)$, for a defining equation $F(T,Y) = 0$. The latter sets $\mathcal{S}'_F$ can be defined for all $F \in \mathcal{P}^{\mathrm{sep}}_{d_1,d_2}$ (not just for those defining Galois covers).

Now, consider the set $\mathcal{S} = \cup_{d_1 \leq D} \cup \{F\} \times \mathcal{S}'_F$, where the inner union is over all $F \in \mathcal{P}^{\mathrm{sep}}_{d_1,|G|}$. From Lemma 3.5 (with $W = \mathcal{P}^{\mathrm{sep}}_{d_1,|G|}$), it follows that $\mathcal{S}$ is contained in a a finite union of varieties, of dimension at most $\dim(\mathcal{P}_{D,|G|}) + R - j + 3 = \delta + R - j + 3 < R$.

Therefore, the image of $\mathcal{S}$ under $\widetilde{\mathrm{PB}}$ is of dimension strictly smaller than R as well. On the other hand, Lemma 3.4 shows that no finite union of varieties of dimension $< R$ can contain defining equations for all k-covers in $\mathsf{H}_{G,R}(\mathbf{C})$.

Hence, $\mathsf{H}_{G,\leq r_0, g\geq 2}(k)$ is not k-regularly parametric. □

Remark 3.7. In fact, the restriction to sets of Galois covers of genus ≥ 2 is used only once in the proof of Theorem 3.1; namely, to ensure that the degree of a rational function T_0, that pulls back a k-cover with $\leq r_0$ branch points to a k-cover in $\mathsf{H}_{G,R}(\mathbf{C})$, is bounded from above in terms of r_0 and R. This then ensures, via the various auxiliary lemmas, that $\mathrm{PB}(\mathsf{H}_{G,\leq r_0,g\geq 2}(k)) \cap \mathsf{H}_{G,R}(\mathbf{C})(k)$ is contained in a finite union of lower-dimensional varieties (as soon as R is sufficiently large), i.e., that its complement inside $\mathsf{H}_{G,R}(\mathbf{C})(k)$ contains a Zariski-dense open subset. If the set $\mathsf{H}_{G,\leq r_0,g\geq 2}(k)$ in Theorem 3.1 is replaced by $\mathsf{H}_{G,\leq r_0,g\geq 1}(k)$, this strong conclusion will no longer be guaranteed. However, since all the auxiliary lemmas remain valid, we obtain in the same way that $\mathrm{PB}(\mathsf{H}_{G,\leq r_0,g\geq 1}(k)) \cap \mathsf{H}_{G,R}(\mathbf{C})(k)$ is contained in a *countable* union of lower-dimensional varieties (each corresponding to rational functions T_0 of some fixed degree). This at least implies $\mathsf{H}_{G,R}(\mathbf{C})(k) \not\subset \mathrm{PB}(\mathsf{H}_{G,\leq r_0,g\geq 1}(k))$ as soon as k is uncountable. Thus the conclusion of Theorem 3.1 remains valid for $\mathsf{H}_{G,\leq r_0,g\geq 1}(k)$ in the important special case $k = \mathbb{C}$.

3.1.4. *Proof of Lemma 3.6.* Let $t_1, \dots, t_s$ be the branch points of f, and let $e_1, \dots, e_s$ be the corresponding orders of inertia groups of f. Set $e_{\max} = \max\{e_1, \dots, e_s\}$. We choose $R_0 = (s-2+j)e_{\max}$, and $R \geq R_0$ arbitrary. We divide the proof into two main steps.

First step: Translation into a combinatorial statement. Let $\sigma_1, \dots, \sigma_s \in S_d$ be the inertia group generators of T_0 at $t_1, \dots, t_s$, and $\sigma_{s+1}, \dots, \sigma_{s+m}$ the non-trivial inertia group generators at further points. For $\sigma \in S_d$, denote by $o(\sigma)$ the number of orbits of $\langle \sigma \rangle$, and set $\mathrm{ind}(\sigma) = d - o(\sigma)$. We claim that, assuming choice of R_0 as above, the following holds:

Claim:

$$\sum_{i=1}^{s} \mathrm{ind}(\sigma_i) \geq 2d - 2 - R + j,$$

or equivalently:

$$\sum_{i=1}^{s} o(\sigma_i) \leq d(s-2) - j + 2 + R.$$

The assertion then follows from the claim, since T_0 defines a genus-zero cover, whence the Riemann–Hurwitz formula yields $\sum_{i=1}^{s+m} \mathrm{ind}(\sigma_i) = 2d - 2$. Together with the claim, this enforces $m \leq R - j$.

Second step: Transformation of cycle structures. To prove the claim, consider the cycle structures of σ_i, $i = 1, \dots, s$. We shall manipulate the cycle structures of the σ_i in a controlled way, to make it easier to estimate the total number of orbits of all σ_i, $i = 1, \dots, s$.

By the definition of T_0 and Abhyankar's lemma, the cycle lengths of the σ_i are multiples of e_i, for $i = 1, \dots, s$, with a total of exactly R exceptions over all $i \in \{1, \dots, s\}$. For each $i \in \{1, \dots, s\}$, let $n_{i,1}, \dots, n_{i,r(i)}$ denote the exceptional cycle lengths (i.e., the ones which are not multiples of e_i), in descending order. Define a permutation $\tau_i \in S_d$ in the following way. First, fill all the non-exceptional cycles of σ_i into τ_i. Next, find the smallest j such that $\sum_{k=1}^{j} n_{i,k} \geq e_i$ and, instead of the cycles of length $n_{i,1}, \dots, n_{i,j}$, fill one single cycle of length $\sum_{k=1}^{j} n_{i,j}$ into τ_i. Repeat this procedure until the sum of the remaining exceptional cycle lengths in σ_i is less than e_i. Fill one more cycle of length the sum of those remaining exceptional cycle lengths into τ_i. By definition, all except possibly one cycle of τ_i have length $\geq e_i$. In particular, the number $o(\tau_i)$ of orbits of $\langle \tau_i \rangle$ is bounded from above by $\lceil d/e_i \rceil < d/e_i + 1$. Also, since f is of genus ≥ 1, the Riemann–Hurwitz formula yields $\sum_{i=1}^{s} \frac{1}{e_i} \leq s - 2$.

Therefore,

$$\sum_{i=1}^{s} o(\tau_i) < d(\sum_{i=1}^{s} \frac{1}{e_i}) + s \leq d(s-2) + s.$$

On the other hand, we can effectively bound the difference between $\sum_{i=1}^{s} o(\tau_i)$ and $\sum_{i=1}^{s} o(\sigma_i)$ via the above construction. Namely, to obtain $(\tau_1, \ldots, \tau_s)$ from $(\sigma_1, \ldots, \sigma_s)$, certain sets of exceptional cycles were replaced by one big cycle. There were R exceptional cycles in total and, since at most $e_{\max}$ cycles each were replaced by one cycle, the total difference $\sum_{i=1}^{s} \big(o(\sigma_i) - o(\tau_i)\big)$ is bounded from above by $R - \frac{R}{e_{\max}}$. Altogether,

$$\sum_{i=1}^{s} o(\sigma_i) \leq R - \frac{R}{e_{\max}} + \sum_{i=1}^{s} o(\tau_i) < R - \frac{R}{e_{\max}} + d(s-2) + s \leq R - (s-2+j) + d(s-2) + s,$$

with the last inequality following from our choice of R_0. This finally yields

$$\sum_{i=1}^{s} o(\sigma_i) \leq d(s-2) - j + 2 + R,$$

showing the claim. This completes the proof.

3.2. **An explicit variant of Theorem 1.2.** The goal of this subsection is to prove the following statement, alluded to in §1.1 as an explicit variant of Theorem 1.2.

Theorem 3.8. *Let k be an algebraically closed field of characteristic 0 and G a finite group with at least 5 maximal non-conjugate cyclic subgroups. Fix an integer $r_0 \geq 0$. For every suitably large even integer R, there is a non-empty Hurwitz stack $\mathsf{H}_{G,R}(\mathbf{C})$ such that* ***no*** *k-cover in $\mathsf{H}_{G,R}(\mathbf{C})$ is a rational pullback of some k-cover in $\mathsf{H}_{G,\leq r_0}$.*

Remark 3.9. Subgroups of $\mathrm{PGL}_2(\mathbb{C})$ have at most 3 maximal non-conjugate cyclic subgroups. Other groups have exactly 3: the quaternion group $\mathbb{H}_8$ or, more generally, dicyclic groups DC_n of order $4n$ $(n \geq 2)$ (which include generalized quaternion groups $\mathrm{DC}_{2^{k-1}}$ $k \geq 2$), groups $\mathrm{SL}_2(\mathbb{F}_q)$ with q a prime power, etc. For these groups, the conclusion from Theorem 1.2 holds but that from Theorem 3.8 is unclear. Replacing 5 by 4 in Theorem 3.8 seems feasible but hard and technical; for the sake of brevity, we avoid this slight improvement. Groups with 4 maximal non-conjugate cyclic subgroups (for which the conclusion from Theorem 3.8 might also hold) include $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, A_6.

Proof. Fix a finite group G with at least 5 non-conjugate maximal cyclic subgroups. At first, assume that not all of them are of order 2. Let $\gamma_1, \ldots, \gamma_5$ be generators of 5 non-conjugate maximal cyclic subgroups of G, and let $\mathcal{C}_1, \ldots, \mathcal{C}_5$ be their conjugacy classes. Denote the order of γ_i by e_i, $i = 1, \ldots, 5$ and, without loss of generality, assume $e_1 > 2$. Consider then a tuple $(\mathcal{C}_1, \ldots, \mathcal{C}_5, \mathcal{C}_6, \ldots, \mathcal{C}_s)$ of conjugacy classes of G, not necessarily distinct, and satisfying the following:

(A) all the non-trivial conjugacy classes of G, but the powers $\mathcal{C}_i^j$, $i = 1, \ldots, 4$, $j = 1, \ldots, e_i - 1$, appear in the set $\{\mathcal{C}_5, \ldots, \mathcal{C}_s\}$.

Consider the $(2s)$-tuple $\underline{\mathcal{C}} = (\mathcal{C}_1, \mathcal{C}_1^{-1}, \ldots, \mathcal{C}_s, \mathcal{C}_s^{-1})$. Note that the integer s can be taken to be any suitably large integer, e.g., by repeating the conjugacy class $\mathcal{C}_5$.

Picking $g_i \in \mathcal{C}_i$ $(i = 1, \ldots, s)$, form the tuple $\underline{g} = (g_1, g_1^{-1}, \ldots, g_s, g_s^{-1})$. As the elements of $\underline{g}$ and their powers contain at least one element from each conjugacy class, a classical lemma of Jordan (see [Jor72]) implies that $\underline{g}$ forms a generating set of G. By construction, the product of entries of $\underline{g}$ is 1. From the Riemann Existence Theorem, $\mathsf{H}_{G,2s}(\underline{\mathcal{C}})(k) \neq \emptyset$.

Let $h \in \mathsf{H}_{G,2s}(\underline{\mathcal{C}})(k)$. Assume there exist $r_0 \in \mathbb{N}$, a Galois cover $f \in \mathsf{H}_{G,\le r_0}(k)$, and $T_0 \in H_{f,k}$ (as defined in §2.2) of degree N such that h and the pullback f_{T_0} are $\mathbb{P}^1_k$-isomorphic. Denote the branch point number of f by r (so $r \le r_0$) and its inertia canonical invariant by $\mathbf{C} = (C_{f,1}, \dots, C_{f,r})$. By [Dèb18, §3.1], the inertia canonical invariant of f_{T_0} is a tuple $\mathbf{C}_{f,T_0}$ obtained by concatenating tuples of the form $\mathbf{C}_{f,T_0,j} = (C_{f,j}^{e_{j1}}, \dots, C_{f,j}^{e_{jr_j}})$, $j = 1, \dots, r$, where $r_j, e_{j,\ell}$ are integers with $r_j \ge 0$ and $e_{j,\ell} \ge 1$ for all $\ell = 1, \dots, r_j$, and $j = 1, \dots, r$. Note that some of the classes in $\mathbf{C}_{f,T_0}$ might be trivial.

Denote by p_j (resp., q_j) the number of $e_{j,\ell}$'s, $\ell = 1, \dots, r_j$, that are equal to 1 (resp., > 1), for $j = 1, \dots, r$. For $j = 1$, denote further by u_1 (resp., v_1) the number of $e_{1,\ell}$'s, $\ell = 1, \dots, r_1$, that are equal to 2 (resp., > 2). Recall further from [Dèb18, §3.1] that, since $e_{j1}, \dots, e_{jr_j}$ are the ramification indices of T_0 over some point, we have $\sum_{\ell=1}^{r_j} e_{j,\ell} = N$, for $j = 1, \dots, r$. Hence, $p_j + 2q_j \le N$, and $p_1 + 2u_1 + 3v_1 \le N$, or equivalently:

$$(1) \qquad q_j \le \frac{N - p_j}{2} \le \frac{N}{2} \text{ for } j = 2, \dots, r \text{ and } v_1 \le \frac{N - p_1 - 2u_1}{3} \le \frac{N}{3}.$$

By the Riemann–Hurwitz formula for T_0, we also have

$$(2) \quad 2N - 2 \ge \sum_{\ell=1}^{r_1} (e_{1,\ell} - 1) + \sum_{j=2}^{r} \sum_{\ell=1}^{r_j} (e_{j,\ell} - 1) = N - p_1 - u_1 - v_1 + \sum_{j=2}^{r} (N - p_j - q_j).$$

As f_{T_0} and h are $\mathbb{P}^1_k$-isomorphic, we have $\mathbf{C}_{f,T_0} = \underline{\mathcal{C}}$, up to order. Without loss of generality, we may assume $\mathcal{C}_1 \in \mathbf{C}_{f,T_0,1}$. For each $i = 1, \dots, 4$, if $j_i \in \{1, \dots, r\}$ is such that $\mathcal{C}_i \in \mathbf{C}_{f,T_0,j_i}$ [11], then, as $\langle \gamma_i \rangle$ is a maximal cyclic subgroup of G, we have $\mathcal{C}_i = C_{f,j_i}^{w_i}$ for some integer w_i relatively prime to e_i. This, together with the assumption that $\mathcal{C}_1, \dots, \mathcal{C}_4$ are not powers of each other, implies that the correspondence $i \mapsto j_i$ is injective. Thus, (2) and (1) give

$$2N - 2 \ge N - p_1 - u_1 - v_1 + \sum_{i=2}^{4} (N - p_{j_i} - q_{j_i}) \ge \frac{13}{6} N - (p_1 + u_1) - \sum_{i=2}^{4} p_{j_i},$$

and hence

$$(3) \qquad N \le 6(p_1 + u_1 + \sum_{i=2}^{4} p_{j_i}).$$

Since each $\mathcal{C}_i$, $i = 1, \dots, 4$, appears at most twice in $\mathbf{C}_{f,T_0}$ by (A), it follows that $p_{j_i} \le 2$ for $i = 1, \dots, 4$. Furthermore, since the powers of $\mathcal{C}_1$ appear at most twice and $e_1 > 2$, we also have $p_1 + u_1 \le 2$. Thus, (3) gives $N \le 48$. However, by a priori choosing s to be large, the degree N of T_0 is forced to be at least 49, contradiction.

It remains to consider the case where all the fixed 5 non-conjugate maximal cyclic subgroups are of order 2. We can reduce to the previous case by changing one of them to another maximal cyclic subgroup of order > 2, unless all elements of G are of order 2. But, in this case, G is an elementary abelian 2-group of rank at least 3, and the number of maximal conjugacy classes of cyclic groups is at least 7. Repeating the above argument with seven classes $\mathcal{C}_1, \dots, \mathcal{C}_7$, replacing the previous $\mathcal{C}_1, \dots, \mathcal{C}_5$, (2) and (1) take the form

$$2N - 2 \ge \sum_{j=1}^{r} (N - p_j - q_j) \text{ and } q_{j_i} \le \frac{N - p_{j_i}}{2} \le \frac{N}{2} \text{ for } i = 1, \dots, 6.$$

[11] There may be *a priori* two different $j_i \in \{1, \dots, r\}$ such that $\mathcal{C}_i \in \mathbf{C}_{f,T_0,j_i}$.

Thus, their combination gives

$$2N-2 \geq \sum_{i=1}^{6}(N-p_{j_i}-q_{j_i}) \geq 3N - \sum_{i=1}^{6} p_{j_i},$$

and so $N \leq \sum_{i=1}^{6} p_{j_i}$. Once again, choosing s and hence N sufficiently large, we obtain a contradiction. □

3.3. **Extension to more general fields.** We have assumed so far that the field k is algebraically closed because we use the Riemann Existence Theorem. However, as we explain below, this assumption can be relaxed in some situations.

Theorem 3.10. *Let G be a finite group and let k be a field of characteristic* 0*.*
(a) *Theorem 1.1*(b) *with $\mathbb{P}^1_{\mathbb{C}}$ replaced by $\mathbb{P}^1_k$ holds in each of these situations:*
- (a-1) *k is ample*[12]*,*
- (a-2) *G is abelian of even order,*
- (a-3) *G is the direct product $A\times H$ of an abelian group A of even order and of a non-solvable group H occuring as a regular Galois group over k* [13]*,*
- (a-4) *$G = S_5$.*

(b) *Theorem 3.8 holds if either k is ample or G is abelian.*

Remark 3.11. The proof below actually gives a little more than what is stated. Under the assumptions of Theorem 3.10(a), not only can we conclude that $\mathsf{H}_G(k) \not\subset \mathrm{PB}(\mathsf{H}_{G,\leq r_0}(k))$ for any integer $r_0 \geq 0$ (as is stated), but what we really obtain is that

$$\mathsf{H}_G(k)\otimes_k \overline{k} \;\not\subset\; \mathrm{PB}(\mathsf{H}_{G,\leq r_0}(\overline{k})).$$

That is: for any $r_0 \geq 0$, one can always find a k-cover in H_G that cannot be obtained, even after base change to $\overline{k}$, by pulling back some $\overline{k}$-cover in $\mathsf{H}_{G,\leq r_0}$ (and even along some non-constant rational map $T_0 \in \overline{k}(U)$ (and not just in $k(U)$). Similarly, if k is ample or G is abelian as in Theorem 3.10(b), what the proof shows is that for any $r_0 \geq 0$ and any suitably large s, there is a Hurwitz stack $\mathsf{H}_{G,2s}(\underline{\mathcal{C}})$ such that $\mathsf{H}_{G,2s}(\underline{\mathcal{C}})(k) \neq \emptyset$ and

$$\mathrm{PB}(\mathsf{H}_{G,\leq r_0}(\overline{k})) \;\cap\; (\mathsf{H}_{G,2s}(\underline{\mathcal{C}})(k)\otimes_k \overline{k}) = \emptyset,$$

and not just $\mathrm{PB}(\mathsf{H}_{G,\leq r_0}(k)) \cap \mathsf{H}_{G,2s}(\underline{\mathcal{C}})(k) = \emptyset$.

3.3.1. *Proof of Theorem 3.10(b).* In the proof of Theorem 3.8 (see §3.2), the assumption $k=\overline{k}$ was only used to guarantee that there is at least one k-cover in the Hurwitz stack $\mathsf{H}_{G,2s}(\underline{\mathcal{C}})$. When k is no longer algebraically closed, we first slightly modify the tuple $\underline{\mathcal{C}}$ to make it *k-rational*, i.e., such that the action of $\mathrm{Gal}(\overline{k}/k)$ on $\underline{\mathcal{C}}$ (taking the power of the classes by the cyclotomic character) preserves $\underline{\mathcal{C}}$, up to the order. This can be done by replacing, for each index $i = 1,\ldots,4$,

each pair $(\mathcal{C}_i, \mathcal{C}_i^{-1})$ by the tuple $(\mathcal{C}_i, \mathcal{C}_i^{-1},,\ldots,\mathcal{C}_i^{e_i-1},\mathcal{C}_i^{-(e_i-1)})$.

The modified tuple is indeed k-rational and the proof of Theorem 3.8 still holds after slight adjustments: the inequalities $p_{j_i} \leq 2$ (resp., $p_1+u_1 \leq 2$) become $p_{j_i} \leq 2(e_i-1)$ (resp., $p_1+u_1 \leq 2(e_1-1)$), $i=1,\ldots,4$, changing only the constants in the final estimates.

With the tuple $\underline{\mathcal{C}}$ now k-rational, it is still true that the Hurwitz stack $\mathsf{H}_{G,2s}(\underline{\mathcal{C}})$ contains a k-cover, so that Theorem 3.8 holds, in the following situations:

[12] Definition of “ample field” is recalled in Remark 1.4(a).

[13] In fact, the assumption on H to be non-solvable can be removed with a bit of extra effort. Moreover, if H is not a regular Galois group over k, then G is not either. Hence, Theorem 1.1(b) trivially fails.

- G abelian and k arbitrary (as a consequence of the classical rigidity theory; see, e.g., [MM18, Chapter I, §4] and [Völ96, §3.2]),
- k ample and G arbitrary. Namely, recall that, over the complete discretely valued field $k((T))$, the so-called 1/2-Riemann Existence Theorem of Pop (see [Pop94]) ensures that $\mathsf{H}_{G,2s}(\underline{\mathcal{C}})(k((T)))$ is non-empty. This implies $\mathsf{H}_{G,2s}(\underline{\mathcal{C}})(k) \neq \emptyset$ for an ample field k, as then k is existentially closed in $k((T))$ (see [Pop96, Proposition 1.1] and [DD97a, §4.2]).

3.3.2. *Proof of Theorem 3.10*(a). In the following, we will denote equivalence classes $[f]$ in $\mathcal{H}_{G,r}(\mathbf{C})$ of a cover f (of group G and ramification type $(r, \mathbf{C})$) simply by f as well when there is no risk of confusion.

We shall deduce Theorem 3.10(a) from Theorem 3.13 below. Start with an arbitrary characteristic 0 field k and consider the following condition, for a finite group G:

Condition 3.12. (a) *There exist a constant $m \geq 0$, infinitely many integers R and, for each R, a ramification type $(R, \mathbf{C})$ for G with the following property: the set of all equivalence classes $[f] \in \mathcal{H}_{G,R}(\mathbf{C})$ such that f is a k-regular Galois cover cannot be covered by finitely many varieties of dimension $< R - m$ (in $\mathcal{H}_{G,R}(\mathbf{C})(\overline{k})$).*
(b) *Each $\mathbf{C}$ in* (a) *contains every conjugacy class of G at least once.*

We then have the following analog of Theorem 3.1 and Lemma 3.2:

Theorem 3.13. *Assume G fulfills Condition 3.12(a) (resp., Condition 3.12(a) and (b)) over a field k of characteristic* 0. *Let $r_0 \in \mathbb{N}$, and let $\mathsf{H} = \mathsf{H}_{G,\leq r_0, g\geq 2}(k)$ (resp., $\mathsf{H} = \mathsf{H}_{G,\leq r_0, g\geq 1}(k)$) be the set of k-regular Galois covers with group G, branch point number $\leq r_0$, and genus ≥ 2 (resp., genus ≥ 1). Then there are infinitely ramification types $(R, \mathbf{C})$ for G such that $\mathsf{H}_{G,R}(\mathbf{C})(k) \not\subset \mathrm{PB}(\mathsf{H})$. In particular, H is not k-regularly parametric.*

Proof. Observe that the crucial Lemma 3.6 in the proof of Theorem 3.1 (see §3.1.3) guarantees this: there exists $R_0 \in \mathbb{N}$ such that, for every ramification type $(R, \mathbf{C})$ for G $(R \geq R_0)$, the set of (defining equations of) Galois covers in $\mathsf{H}_{G,R}(\mathbf{C})(\overline{k})$ which arise as rational pullbacks of some cover with $\leq r_0$ branch points is contained in a union of finitely many varieties of dimension at most $R - (m+1)$. Then, with R sufficiently large and $(R, \mathbf{C})$ as in Condition 3.12, it follows as in Lemma 3.4 that these varieties are not sufficient to yield defining equations for every cover in $\mathsf{H}_{G,R}(\mathbf{C})(\overline{k})$ which is defined over k. Furthermore, the additional Condition 3.12(b) is sufficient for the proof of Lemma 3.2 over k, if G is non-cyclic. For cyclic $G = \mathbb{Z}/n\mathbb{Z}$, Lemma 3.2 holds as soon as the genus-0 extension $k(\sqrt[n]{T})/k(T)$ is k-regular, i.e., as soon as $e^{2i\pi/n} \in k$. On the other hand, Galois groups of genus-1 cyclic covers are only $\mathbb{Z}/n\mathbb{Z}$ for $n \in \{2, 3, 4, 6\}$, and it is easy to verify that, for $e^{2i\pi/n} \notin k$, these genus-1 covers cannot be defined over k either, as a special case of the *Branch Cycle Lemma* (see [Fri77] and [Völ96, Lemma 2.8]). □

Proof of Theorem 3.10(a-1). Let k be an ample field of characteristic zero. Equivalently to the definition of "ample", every k-variety with a simple k-rational point has a Zariski-dense set of k-rational points (see [Jar11, Lemma 5.3.1]). On the other hand, the 1/2-Riemann Existence Theorem yields plenty of class r-tuples $\mathbf{C}$ of G with $\mathsf{H}_{G,r}(\mathbf{C})(k) \neq \emptyset$. For example, all $\mathbf{C}$ corresponding to an arbitrarily long repetition of the tuple $(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1})$, where x_i runs through all non-identity elements of G, are fine. For $Z(G) = \{1\}$, this then implies the existence of a k-rational point on $\mathcal{H}_{G,r}(\mathbf{C})$ (cf. §2.1.2) and as k is ample that at least one connected component of $\mathcal{H}_{G,r}(\mathbf{C})$ has a Zariski-dense set of k-rational points. Since these k-rational points are equivalence classes of covers f defined over k and all components are of the same dimension, Condition 3.12 holds, even with $m = 0$, and, therefore, Theorem 3.13 holds over k. Finally, the case of arbitrary G

can be reduced to the above assumption $Z(G) = \{1\}$ by elementary means, using that every finite group is a quotient of a group with trivial center. Concretely, for a group G of order n, choose any non-abelian simple group S and consider the wreath product $\Gamma := S \wr G = S^n \rtimes G$. Then Γ clearly has trivial center. Furthermore, the conjugates of a given complement $G \leq \Gamma$ of S^n generate all of Γ. Therefore, if $(r, \mathbf{C})$ is a ramification type of G with $\mathcal{H}_{G,r}(\mathbf{C})(\overline{k}) \neq \emptyset$, then for a suitable multiple R of r, there exists a ramification type $(R, \tilde{\mathbf{C}})$ of Γ projecting onto a repetition of $\mathbf{C}$, and such that $\mathcal{H}_{\Gamma,R}(\tilde{\mathbf{C}})(k) \neq \emptyset$. As above, the equivalence classes of k-regular covers of group Γ and ramification type (any arbitrarily long repetition of) $\tilde{\mathbf{C}}$ are Zariski-dense in at least one connected component, thus fulfilling Condition 3.12. But by the choice of $\tilde{\mathbf{C}}$, for any such cover, the subcover with group G is a k-regular cover with the same branch point set, thus yielding Condition 3.12 also for G. □

Proof of Theorem 3.10(a-2). As a first example, let G be an elementary abelian 2-group. Since G is abelian, every tuple $(C_1, \ldots, C_R)$ of conjugacy classes in G with non-empty Nielsen class is a *rigid* tuple (in the sense of, e.g., [MM18, Chapter I, §4]).

Furthermore, since all non-identity elements of G are of order 2, every conjugacy class is trivially *rational* (i.e., unchanged if taken to a power relatively prime to the order of its elements). It then follows from the rigidity method that, for every choice $(t_1, \ldots, t_R)$ of R distinct points in $\mathbb{P}^1(k)$, there exists a Galois cover of $\mathbb{P}^1$, defined over k, with inertia canonical invariant $(C_1, \ldots, C_R)$ and branch points $(t_1, \ldots, t_R)$. In particular, the set of all these covers cannot be obtained by a set of defining equations of dimension $< R$. Hence, G fulfills Condition 3.12 over k. The assertion of Theorem 3.13 therefore holds for G over an arbitrary field of characteristic zero.

Now, let G be an arbitrary abelian group of even order. As before, all class tuples with non-empty Nielsen class are rigid. Let $(C_1, \ldots, C_R)$ be a class tuple of G containing each element of order ≥ 3 exactly once, and each element of order 2 an arbitrary even number of times. This then yields a product-1 tuple generating G and, if the branch points for each set of generators of a cyclic subgroup $\mathbb{Z}/d\mathbb{Z}$ are chosen appropriately to form a full set of conjugates (under the action of $\mathrm{Gal}(\mathbb{Q}(e^{2i\pi/d})/\mathbb{Q})$) in $k(e^{2i\pi/d})$, then the associated ramification type is a rational ramification type, implying that there is again a k-regular Galois cover with inertia canonical invariant $(C_1, \ldots, C_R)$ and with the prescribed branch point set. Note that the branch points for elements of order 2 are still allowed to be chosen freely in k. Since there are less than $|G|$ other branch points, it follows as above that the set of Galois covers with these ramification data cannot be obtained by a set of defining equations of dimension $\leq R - |G|$. Therefore again, Condition 3.12 is fulfilled. □

Proof of Theorem 3.10(a-3). Let $G = A \times H$, where A is an abelian group of even order and H is any non-solvable group which occurs as a regular Galois group over k. We use the non-solvability assumption only to obtain that there are no Galois covers of genus ≤ 1 with group G. Take a tuple $(C_1, \ldots, C_R)$ of classes of $A \leq G$ as in the proof of Theorem 3.10(a-2), and prolong it by a fixed tuple $(C_{R+1}, \ldots, C_S)$ of classes which occurs as some ramification type for H over k. With the appropriate choice of branch point set for the H-cover, we obtain a k-regular Galois cover with group $A \times H$ where, once again, the branch points with involution inertia in A can be chosen freely (outside of the fixed branch points of the H-cover). Increasing R as in the proof of Theorem 3.10(a-2) (whilst fixing $(C_{R+1}, \ldots, C_S)$), we again obtain that Condition 3.12(a) is fulfilled, and so Theorem 3.13 holds over k for Galois covers of genus ≥ 2. As there is no Galois cover of group G and genus ≤ 1, Theorem 1.1(b) holds for G over all fields of characteristic 0. □

Proof of Theorem 3.10(a-4). Let $M_{g,d}$ be the moduli space of simply branched covers (i.e., all non-trivial inertia groups are generated by transpositions) of degree d and genus g. It is known (see [AC81]) that $M_{g,5}$ is unirational for all $g \geq 6$ and, in fact, this holds even over the smallest field of definition $\mathbb{Q}$ (and, hence, over all fields of characteristic zero).

Note that $M_{g,5}$ parameterizes Galois covers in $\mathsf{H}_{S_5,8+2g}(\mathbf{C})$ with $\mathbf{C} = (C_1, \dots, C_{8+2g})$, where each C_i is the class of transpositions. This space is of dimension $8+2g$, and unirationality (over k) implies that its function field is of finite index in some $k(T_1, \dots, T_{8+2g})$ with independent transcendentals T_i. But, of course, every k-rational value of $(T_1, \dots, T_{8+2g})$ then leads to a k-rational point on $M_{g,5}$ (and, thus, a cover defined over k), and the set of such k-rational points on a unirational variety is always Zariski-dense.

This implies that Condition 3.12(a) is fulfilled and, as there is no Galois cover of group S_5 and genus ≤ 1, Theorem 1.1(b) holds for S_5 over all fields of characteristic 0. □

4. Proof of Theorem 1.3

Throughout this section, fix an algebraically closed field k of characteristic 0 and a Galois cover $f : X \to \mathbb{P}^1_k$ with group G and ramification type $(R, \mathbf{C})$ for $R \geq 4$. Assume $G \not\subset \mathrm{PGL}_2(\mathbb{C})$. Let $\mathbf{t} = \{t_1, \dots, t_R\}$ be the branch point set of f. Let $e_f(t_0)$ denote the ramification index of $t_0 \in \mathbb{P}^1(k)$ under f, and set $e_i = e_f(t_i)$ for $i = 1, \dots, R$. Given $T_0 \in k(U) \setminus k$ and $t_0 \in \mathbb{P}^1(k)$, let $e(q|t_0)$ denote the ramification index under T_0 of $q \in T_0^{-1}(t_0)$. The proof is based on the following estimate on the branch point number of a pullback, which strengthens the bounds in [Dèb18, Theorem 3.1(b-2)]. Recall that for $T_0 \in H_{f,k}$, the fiber product $X \times_{f,T_0} \mathbb{P}^1_k$ is irreducible, cf. Section 2.2.

Lemma 4.1. *Let $T_0 \in H_{f,k}$ be of degree n. Let a_i be the number of preimages $q \in T_0^{-1}(t_i)$ with $e_i \mid e(q|t_i)$ for $i = 1, \dots, R$, and $U_{T_0,f}$ the set of points $q \in \mathbb{P}^1(k)$ such that $e(q|t_0) \neq e_f(t_0)$ for $t_0 = T_0(q)$. Then the branch point number R_{T_0} of the pullback f_{T_0} is at least*

$$R_{T_0} \geq (R-4)n + 4 + \sum_{i=1}^{R} (e_i - 2)a_i + \sum_{q \in U_{T_0,f}} \big(e(q|t_0) - 1\big).$$

Moreover, equality holds if and only if T_0 is unramified away from $\mathbf{t}$ and its ramification indices over t_i are either e_i or not divisible by e_i, for $i = 1, \dots, R$.

Proof. Let b_i denote the number of preimages q in $T_0^{-1}(t_i)$ such that $e(q|t_i)$ is not divisible by e_i, for $i = 1, \dots, R$. Note that, since $X \times_{f,T_0} \mathbb{P}^1_k$ is irreducible, Abhyankar's lemma implies

$$R_{T_0} = b_1 + \cdots + b_R. \tag{4}$$

By the Riemann–Hurwitz formula for T_0, we have

$$2n - 2 = \sum_{t_0 \in \mathbb{P}^1} \sum_{q \in T_0^{-1}(t_0)} (e(q|t_0) - 1) \geq \sum_{q \in U_{T_0,f}} \big(e(q|t_0) - 1\big) + \sum_{i=1}^{R} (e_i - 1)a_i, \tag{5}$$

with equality if and only if $e(q|t_i)$ is either e_i or non-divisible by e_i, for all points $q \in T_0^{-1}(t_i), i = 1, \dots, R$. The same Riemann–Hurwitz formula also implies

$$2n - 2 \geq \sum_{i=1}^{R} (n - a_i - b_i) = Rn - \sum_{i=1}^{R} (a_i + b_i),$$

with equality if and only if T_0 is unramified away from $\mathbf{t}$. Combined with (5), this gives

$$\begin{aligned}2n-2 &\geq Rn - \sum_{i=1}^{R}(a_i+b_i)\\ &\geq Rn + \sum_{i=1}^{R}(e_i-2)a_i + \sum_{q\in U_{T_0,f}}\big(e(q|t_0)-1\big) - (2n-2) - \sum_{i=1}^{R} b_i.\end{aligned}$$

Combined with (4), this gives

$$R_{T_0} = \sum_{i=1}^{R} b_i \geq (R-4)n+4+\sum_{i=1}^{R}(e_i-2)a_i + \sum_{q\in U_{T_0,f}}\big(e(q|t_0)-1\big),$$

with equality if and only if T_0 is unramified away from $\mathbf{t}$ and every ramification index $e(q|t_i)$, for $q\in T_0^{-1}(t_i)$, $i=1,\dots,R$, which is divisible by e_i is equal to e_i. □

Let E denote the multiset $\{e_1,\dots,e_R\}$. Throughout the proof below, we assume T_0 is in the Hilbert subset $H_{f,k}$, and hence that the fiber product $X\times_{f,T_0}\mathbb{P}^1_k$ is irreducible.

Proof of Theorem 1.3 when $E\neq\{2,2,2,3\},\{2,2,2,4\}$. The case where f is of genus 1 follows from Remark 3.3, so henceforth we shall assume that the genus of X is at least 2. Let $(g_1,\dots,g_R)$ be a tuple in the Nielsen class of $\mathbf{C}$, corresponding to f. As any permutation of the tuple $\mathbf{C}=(C_1,\dots,C_R)$ has a non-empty Nielsen class, without loss of generality, we may assume that g_i is a branch cycle over t_i and the orders $e_1,\dots,e_R$ of $g_1,\dots,g_R$ are ordered in decreasing order. This tuple can be modified to a tuple (P_y) $y, y^{-1}g_1,\dots,g_R$ for any $y\neq g_1$. Such a tuple generates G and has product 1, giving a non-empty Nielsen class corresponding to a ramification type which we denote by $(R+1,\mathbf{D}_y)$. We will show that, for a suitable choice of $y\in G$, no pullback f_{T_0}, along $T_0\in k(U)\setminus k$ of degree $n>1$, is in $\mathsf{H}_{G,R+1}(\mathbf{D}_y)(k)$. Since a cover in $\mathsf{H}_{G,R+1}(\mathbf{D}_y)(k)$ is a pullback of f only if y is a power of some element in $C_1,\dots,C_R$, by varying y, we may assume that every conjugacy class in G is a power of one of $C_1,\dots,C_R$.

As in Lemma 4.1, let a_i (resp., b_i) be the number of preimages q in $T_0^{-1}(t_i)$ such that the ramification index $e(q|t_i)$ is divisible by e_i (resp., is not divisible by e_i) for $i=1,\dots,R$. For $R\geq 6$, as $n\geq 2$, the lower bound on R_{T_0} from Lemma 4.1 is $\geq R+2$, as desired[14].

The case $R=5$: We may assume $R_{T_0}=6$. By Lemma 4.1, we have

$$n\leq 2-\sum_{i=1}^{R}(e_i-2)a_i - \sum_{q\in U_{T_0,f}}\big(e(q|t_0)-1\big).$$

As $n>1$, this forces $n=2$, $e(q|t_0)=1$ for all $q\in U_{T_0,f}$, and $\sum_{i=1}^{R}(e_i-2)a_i=0$. Thus, every ramification index of T_0 over t_i is either e_i or 1, and T_0 is unramified away from $\mathbf{t}$. Since T_0 is of degree 2, it ramifies over exactly two branch points with ramification index 2. Hence, without loss of generality, we may assume $e_4=e_5=2$, so that the inertia canonical invariant of f_{T_0} is $\mathbf{C}_{T_0}=(C_1,C_1,C_2,C_2,C_3,C_3)$. If $e_1>2$, by picking y to be an involution, no cover in $\mathsf{H}_{G,R+1}(\mathbf{D}_y)(k)$ is a pullback of f, as $\mathbf{D}_y$ has more conjugacy classes of involutions than $\mathbf{C}_{T_0}$ does. Similarly, if $e_1=2$ and G contains an element y of order >2, then every cover in $\mathsf{H}_{G,R+1}(\mathbf{D}_y)(k)$ is not a pullback of f, as its Nielsen class has a conjugacy class of non-involutions. If $e_1=2$ and all elements of G are of order 2,

[14]We note that, for $R\geq 6$, this claim also follows for any choice of y from [Dèb18, Theorem 3.1(b-2)], since the latter implies that every pullback of a k-cover in $\mathsf{H}_{G,R}(\mathbf{C})$ has at least $R+2$ branch points, and hence is not in $\mathsf{H}_{G,R+1}(\mathbf{D}_y)(k)$.

then G is an elementary abelian 2-group generated by three conjugacy classes C_1, C_2, C_3. As $G \not\subset \mathrm{PGL}_2(\mathbb{C})$, this forces $G \cong (\mathbb{Z}/2\mathbb{Z})^3$. In the latter case, we may choose y to be an involution whose conjugacy class is different from C_1, C_2, C_3, C_4, C_5, so that no cover in $\mathsf{H}_{G,R+1}(\mathbf{D}_y)(k)$ is a pullback of f.

The case $R = 4$: Assume $R_{T_0} = 5$. By Lemma 4.1, we have

$$\sum_{i=1}^{R}(e_i - 2)a_i + \sum_{q\in U_{T_0,f}} \big(e(q|t_0) - 1\big) \leq 1. \tag{6}$$

Let $\mathbf{t}_2$ (resp., $\mathbf{t}_{>2}$) denote the set of t_i with $e_i = 2$ (resp., $e_i > 2$). Since every point $q \in T_0^{-1}(t_i)$, $t_i \in \mathbf{t}_2$, with ramification index $e(q|t_0) > 2$ contributes at least 2 to (6), we deduce that $e(q|t_i) = 1$ or 2 for all $q \in T_0^{-1}(t_i)$, $t_i \in \mathbf{t}_2$. On the other hand, for $t_i \in \mathbf{t}_{>2}$, (6) gives $a_i = 0$ with the possible exception of a single t_ι for which $a_{t_\iota} = 1$ and $e_\iota = 3$. Moreover, if such ι exists, then the contribution of the sum over $U_{T_0,f}$ in (6) is 0. Hence,

(7) $e(q|t_i) = 1$ for all points $q \in T_0^{-1}(t_i)$, $t_i \in \mathbf{t}_{>2}$, with the possible exception of a single $q_\iota \in T_0^{-1}(t_\iota)$ where $e_\iota = e(q_\iota|t_\iota) = 3$.

Otherwise, $a_i = 0$ for all $t_i \in \mathbf{t}_{>2}$, in which case (6) shows that

(8) $e(q|t_i) = 1$ for all points $q \in T_0^{-1}(t_i)$, $t_i \in \mathbf{t}_{>2}$, with the possible exception of a single $q_\eta \in T_0^{-1}(t_\eta)$ where $e(q_\eta|t_\eta) = 2 \neq e_\eta$.

If G is of odd order, (7) and (8) force T_0 to have at most one ramification point, which, by the Riemann–Hurwitz formula, forces a contradiction to $n > 1$. Henceforth, assume G is of even order.

Let $a = |\mathbf{t}_{>2}|$. Since the genus of X is more than 1 and $R = 4$, the Riemann–Hurwitz formula for f implies $a \geq 1$. Assume first $a > 1$. In this case, we let $y \in G$ be an involution. The number of elements of order > 2 in (P_y) is a or $a - 1$. In case there exists ι as above, (7) forces every point $t_i \in \mathbf{t}_{>2} \setminus \{t_\iota\}$ to have at least three unramified preimages and hence forces f_{T_0} to have at least $3(a-1)$ branch points with ramification index > 2. As $3(a-1) > a$ for $a > 1$, the pullback f_{T_0} does not not have ramification type $(R+1, \mathbf{D}_y)$. The same argument applies if (8) holds and $n \geq 3$. If $n = 2$ and (8) holds, then f_{T_0} has at least $2(a-1)+1$ branch points with ramification index > 2. Similarly, $2(a-1)+1 > a$, and hence f_{T_0} does not have ramification $(R+1, \mathbf{D}_y)$.

Henceforth, assume $a = 1$, that is, $E = \{e, 2, 2, 2\}$. Note that $e > 4$ by assumption. Condition (7) does not hold since $e > 3$, and hence (8) does. The latter implies that the inertia canonical conjugacy classes of f_{T_0} over points in $T_0^{-1}(t_1)$ are either C_1 or C_1^2, hence of order e or $e/2$. Thus, if we pick y to be of order different from e and $e/2$, then the only element of (P_y) that can appear in such conjugacy class is yx_1. Thus, (8) implies:

(9) $T_0^{-1}(t_1)$ consists of a single point q_η with ramification $e(q_\eta|t_1) = 2$ such that the inertia canonical invariant of f_{T_0} over q_η is the conjugacy class of yx_1.

The latter then has to be of order $\widetilde{e} = e/2$ or e.

At first, consider the case where the conjugacy classes C_2, C_3, C_4 do not coincide. Without loss of generality, we may then assume that C_2 is different from C_3 and C_4. Picking $y = x_3$, (9) implies that $n = 2$, and $(R+1, \mathbf{D}_y)$ is a ramification type consisting of conjugacy classes of orders $2, \widetilde{e}, 2, 2, 2$, with $\widetilde{e} = e$ or $e/2$. In particular, $\widetilde{e} > 2$ and C_2 appears at most once in $\mathbf{D}_y$. Since $n = 2$ and f_{T_0} is assumed to have 5 branch points, T_0 has to ramify over exactly one of the places t_2, t_3, t_4, say t_k. If $k = 2$, then C_2 does not appear in the ramification type of f_{T_0}, contradicting its appearance in $\mathbf{D}_y$. If $k = 3$ or

4, then t_2 is unramified under T_0 and hence C_2 appears at least twice in the ramification type of f_{T_0}, but only once in $\mathbf{D}_y$, contradiction.

Now, consider the case $C_2 = C_3 = C_4$. Note that, since G is non-cyclic, C_2 is not a power of C_1. Assume next that e is even. We may then pick y to be an involution which is a power of x_1. As y is not of order e or $e/2$, we get $y \notin C_1 \cup C_1^2 \cup C_2$, contradicting that those are the only conjugacy classes that may appear in the ramification type of f_{T_0}. Next, assume e is odd, and pick a prime p dividing e. If $p = e$, then G is solvable by Burnside's theorem. Letting N be a minimal normal subgroup of G, it follows that N contains exactly one of C_1 and C_2. The product 1 relation then gives a contradiction in G/N. Thus, we may assume p is a proper divisor of e. In this case, if we pick y to be a power of x_1 of order p, then (9) implies that the only conjugacy classes appearing in the ramification type of f_{T_0} are C_1, C_1^2 or C_2, neither of which contains y, contradiction. This concludes the proof in the case $E \neq \{2,2,2,3\}, \{2,2,2,4\}$. □

Remark 4.2. In the following, we shall use Magma for computations with small order groups. More specifically, we use the command ExtensionsOfElementaryAbelianGroup to run over extensions of a given group by an elementary abelian group.

Finally, we show that, if E is $\{3,2,2,2\}$ or $\{4,2,2,2\}$, then either $G \subseteq \mathrm{PGL}_2(\mathbb{C})$ or there is no group G whose maximal conjugacy classes are the classes of a product 1 tuple $x_1, \ldots, x_4$ with orders in E. Since, in both cases, $|G|$ is divisible by at most two primes, G is solvable by Burnside's theorem. Let N be a minimal normal subgroup of G, so that $N \cong (\mathbb{Z}/2\mathbb{Z})^u$ or $(\mathbb{Z}/3\mathbb{Z})^u$, for $u \geq 1$.

Case {3,2,2,2}: If $N \cong (\mathbb{Z}/3\mathbb{Z})^u$, the images of x_2, x_3, x_4 in G/N remain of order 2 and hence $G/N \cong (\mathbb{Z}/2\mathbb{Z})^2$. Since, moreover, G/N acts transitively on the $\mathbb{Z}/3\mathbb{Z}$ subgroups of N, it follows that $u = 1$ or 2. A check using Magma (see Remark 4.2) shows that all such group extensions G contain an element of order 6.

In the case $N \cong (\mathbb{Z}/2\mathbb{Z})^u$, the subgroup N contains exactly one of the conjugacy classes C_2, C_3, C_4 since, otherwise, we get a contradiction to the product 1 relation in G/N. Since G/N is generated by three elements of orders $3, 2$ and 2 with product 1, it is isomorphic to S_3. As G/N acts transitively on the $\mathbb{Z}/2\mathbb{Z}$ copies in N, we have $u = 1$ or 2. Once again, a Magma check shows that all such group extensions G contain an element of order 4 or 6, contradicting that C_i, $i = 1, \ldots, 4$ are the only maximal ones.

Case {4,2,2,2}: In this case, G is a 2-group, and $N \cong (\mathbb{Z}/2\mathbb{Z})^u$. The conjugacy classes of involutions in G are C_1^2, C_2, C_3, and C_4. If N does not contain C_1^2, then the product one relation in G/N implies that N contains exactly one of the conjugacy classes C_2, C_3, C_4. In this case, G/N is dihedral of order 8, acting transitively on the $\mathbb{Z}/2\mathbb{Z}$ copies in N. Hence, $u = 1$ or 2. A Magma check (see Remark 4.2) shows that such a G either contains an element of order 8, or has more than four conjugacy classes of involutions, or has more than two conjugacy classes of elements of order 4 (in which case there is more than one conjugacy class of cyclic subgroups of order 4).

If N contains C_1^2, then all elements of G/N are involutions, and hence G/N is an elementary abelian 2-group. Since we may assume N is a proper subgroup of G, the product 1 relation in G/N implies that N contains one or two of the conjugacy classes C_2, C_3, C_4. In the former case, G/N is a 2-group acting transitively on involutions in N, forcing $u = 1$. In this case, a Magma check shows that, for such group extensions, either the number of conjugacy classes of involutions is more than 4 or the number of conjugacy classes of order 4 elements is more than 2. If N contains two of C_2, C_3, C_4, then $G/N \cong \mathbb{Z}/2\mathbb{Z}$. Thus, by minimality of N, we get that $u \leq 2$ and hence that G is isomorphic to a subgroup of $\mathrm{PGL}_2(\mathbb{C})$.

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Email address: `pierre.debes@univ-lille.fr`

Email address: `jkoenig@knue.ac.kr`

Email address: `francois.legrand@tu-dresden.de`

Email address: `dneftin@technion.ac.il`

Laboratoire de Mathématiques Paul Painlevé, Université de Lille, 59655 Villeneuve d'Ascq Cedex, France

Department of Mathematics Education, Korea National University of Education, 28173 Cheongju, South Korea

Institut für Algebra, Fachrichtung Mathematik, TU Dresden, 01062 Dresden, Germany

Department of Mathematics, Technion, Israel Institute of Technology, Haifa 32000, Israel